# Linear Quadratic Optimal Control for (In)-Finite Discrete Time Systems

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## Contents



## 1 Notation

Let $A, B, C$ be sets. The function arrow is right associative. That is to say $f : A \to B \to C$ has to be parsed as $f : A \to (B \to C)$. Meaning that $f$ is a function from $A$ to the set of all functions with domain $B$ and codomain $C$.
Let $a \in A, b \in B, c \in C$. Function application is denoted by a white space. That is to say $f\ a$ has to be parsed as $f$ applied to $a$. The white space is also sometimes removed, in the case of two brackets () wrapping the thing the function is applied to. Function application is left associative. That is to say that $f\ a\ b\ c$ has to be parsed as $(((f\ a)\ b)\ c)$.
The \$ sign denotes infix function application. The infix priority of \$ is set such that it is parsed after equal signs and before all other infix operators. For example the expression

$$3 = A \circ B \ \$\ v + 3 \cdot w, \tag{1}$$

where $A, B, v, w$ are such that the expression makes sense, has to be parsed as

$$3 = (A \circ B)\ (v + 3 \cdot w). \tag{2}$$

## 2 Finite Time Case

Throughout this section let $X, Y, U$ Hilbert spaces and $A \in L(X), B \in L(U, X), C \in L(X, Y), D \in L(U, Y)$. Let $L_{\geq 0}(X)$ be the set of all symmetric and non-negative linear operators $X \to X$. Let

$$S := \{(k, N) \in \mathbb{N}^2 : k < N\} \tag{3}$$

and $\operatorname{range} : \mathbb{N} \to \mathbb{N} \to \mathcal{P}\ \mathbb{N}$

$$\operatorname{range} n\ m := \{k \in \mathbb{N} : n \leq k \leq m\}. \tag{4}$$

**Definition 1 (Solution map).** Define $\operatorname{soln} : S \to \operatorname{Mor} \mathbf{Set}$ ($\operatorname{Mor} \mathbf{Set} =$ the "set" of all functions between sets) by

$$\begin{aligned} \operatorname{soln}(k, N) &: \operatorname{range} k\ N \to X \to (\operatorname{range} k\ (N-1) \to U) \to (\operatorname{range} k\ N \to X) \\ \operatorname{soln}(k, N)\ x_0\ u\ k &:= x_0, \\ \operatorname{soln}(k, N)\ x_0\ u\ (n+1) &:= A\ (\operatorname{soln}(k, N)\ x_0\ u\ n) + B(u\ n) \end{aligned} \tag{5}$$

**Proposition 2.** *Let* $(k, N) \in S, x_0 \in X$ *and* $u_1, u_2 \in \operatorname{range} k(N-1) \to U$*. Let* $x_1 := \operatorname{soln}(k, N)\ x_0\ u_1$*,* $x_2 := \operatorname{soln}(k, N)\ x_0\ u_2$ *and* $x := \operatorname{soln}(k, N)\ 0\ (u_1 - u_2)$*. Then*

$$x_1 - x_2 = x. \tag{6}$$

*Proof.* By induction:

$$x_1\ k - x_2\ k = x_0 - x_0 = 0 = x\ k. \tag{7}$$

Assume that the assertion in the proposition is true for $n \in \operatorname{range} k\ (N-1)$. Then

$$\begin{aligned} x_1\ (n+1) - x_2\ (n+1) &= A(x_1\ n) + B(u_1\ n) - A(x_2\ n) - B(u_2\ n) \\ &= A(x_1\ n - x_2\ n) + B(u_1\ n - u_2\ n) \\ &= A\ (x\ n) - B((u_1 - u_2)\ n) = x(n+1). \end{aligned} \tag{8}$$

□

**Proposition 3.** *Let* $(k, N) \in S$*. Then* $T : \ell^2(\operatorname{range} k\ (N-1), U) \to \ell^2(\operatorname{range} k\ N, X)$ *defined by* $T := \operatorname{soln}(k, N)\ 0$ *is linear and bounded.*

*Proof.* Let $u \in \ell^2(\operatorname{range} k\ (N-1), U)$ and $x := T\ u$. Then for all $n \in \operatorname{range}(k+1)\ N$:

$$x\ n = \sum_{j=0}^{n-k-1} A^j(B\ u(n-j-1)). \tag{9}$$

The proof is by induction:

$$x\ (k+1) = A(x\ k) + B(u\ k) = B(u\ k). \tag{10}$$

Let $n \in \operatorname{range}(k+1)\ (N-1)$ and assume that the above equation for $x\ n$ is true. Then

$$\begin{aligned} x\ (n+1) &= A\ (x\ n) + B\ (u\ n) \\ &= \sum_{j=0}^{n-k-1} A^{j+1}(B\ u(n-j-1)) + B(u\ n) \\ &= \sum_{j=1}^{(n+1)-k-1} A^j(B\ u((n+1)-j-1)) + B(u\ n) \\ &= \sum_{j=0}^{(n+1)-k-1} A^j(B\ u((n+1)-j-1)). \end{aligned} \tag{11}$$

Since $A$ and $B$ and the evaluation maps are linear and bounded it follows that $p_i \circ T$, where $p_i$ is the $i$-th coordinate projection, is linear and bounded for all $i \in \{1, \dots, N\}$, which in turn implies the linearity and the boundedness of $T$. □

**Definition 4 (Output map).** Define $\operatorname{out} : S \to \operatorname{Mor} \mathbf{Set}$ by

$$\begin{aligned} \operatorname{out}(k, N) : X \to (\operatorname{range} k\ (N-1) \to U) \to (\operatorname{range} k\ N \to Y) \\ \operatorname{out}(k, N)\ x_0\ u := C \circ (\operatorname{soln}(k, N)\ x_0\ u) + D \circ u. \end{aligned} \tag{12}$$

**Definition 5 (Cost functional).** Define $J : L_{\geq 0}(X) \to S \to \operatorname{Mor} \mathbf{Set}$ by

$$\begin{aligned} J\ P_0\ (k, N) : X \to (\operatorname{range} k\ (N-1) \to U) \to [0, \infty) \\ J\ P_0\ (k, N)\ x_0\ u := \sum_{n=k}^{N-1} \left(\|y\ n\|^2 + \|u\ n\|^2\right) + \langle (x\ N, P_0(x\ N)\rangle, \\ \text{where } y := \operatorname{out}(k, N)\ x_0\ u,\ x := \operatorname{soln}(k, N)\ x_0\ u. \end{aligned} \tag{13}$$

In the remainder of this subsection let $N \in \mathbb{N}$ be fixed.

**Proposition 6 (Existence and uniqueness of minimizer).** *For all* $P_0 \in L_{\geq 0}(X), k \in \operatorname{range} 1\ (N-1), x_0 \in X : J\ P_0\ (k, N)\ x_0$ *has a unique minimizer.*

*Proof.* Let $P_0 \in P_0 \in L_{\geq 0}(X)$ and $k \in \operatorname{range} 1\ (N-1)$. Let $H := X \times \ell^2(\operatorname{range} k\ (N-1), Y) \times \ell^2(\operatorname{range} k\ (N-1), U)$. Define $V : X \to \mathcal{P}\ H$ by

$$V\ x_0 := \{(x, y, u) \in H : x = \sqrt{P_0}(\operatorname{soln} k\ x_0\ u\ N),\ y = \operatorname{out} k\ x_0\ u\}. \tag{14}$$

Then for all $x_0 \in X$ finding a minimum of $J\ P_0\ (k, N)\ x_0$ is the same as finding a minimum of the norm squared on $V\ x_0 \neq \varnothing$.
Now $V\ 0$ is a closed subspace of $H$, because it is the graph of a bounded linear operator (follows from proposition 3).
Furthermore $\forall x_0 \in X : V\ x_0 = v + V\ 0$, where $v \in V\ x_0$ arbitrary (follows from proposition 2). Let $P$ be the orthogonal projection onto the orthogonal complement of $V\ 0$. Then, by a standard result, $V\ x_0$ contains a unique element with minimal norm given by $P\ v_0$ where $v_0 \in V\ x_0$ is arbitrary. □

**Definition 7 (Minimizer map).** Define $M : L_{\geq 0}(X) \to \operatorname{range} 1\ (N-1) \to \operatorname{Mor} \mathbf{Set}$ by letting $M\ P_0\ k : X \to (\operatorname{range} k\ (N-1) \to U)$ be the map that sends $x_0$ to the unique minimizer of $J\ P_0\ (k, N)\ x_0$.

**Proposition 8 (Bellmans principle of optimality).** *Let $x_0 \in X, P_0 \in L_{\geq 0}(X)$ and $k \in \operatorname{range} 1\ (N-1)$. Let $u_0 := M\ P_0\ 1\ x_0$ and $u_1 := M\ P_0\ k\ (\operatorname{soln}(1, N)\ x_0\ u_0\ k)$. Then for all $n \in \operatorname{range} k\ (N-1)$:*

$$u_0\ n = u_1\ n. \tag{15}$$

*Proof.* Let $k \in \operatorname{range} 1\ (N-1)$ and $u \in \operatorname{range} 1\ (N-1) \to U$. Let $y := \operatorname{out}(1, N)\ x_0\ u,\ x := \operatorname{soln}(1, N)\ x_0\ u$. Then

$$\begin{aligned} J\ P_0\ (1, N)\ x_0\ u &= \sum_{n=1}^{N-1} \left(\|y\ n\|^2 + \|u\ n\|^2\right) + \langle (x\ N, P_0(x\ N) \rangle \\ &= \sum_{n=1}^{k-1} \left(\|y\ n\|^2 + \|u\ n\|^2\right) + J\ P_0\ (k, N)\ (x\ k)\ u|_{\ldots} \end{aligned} \tag{16}$$

The first term and the value of $x\ k$ is independent of the values of $u$ for indices strictly larger than $k-1$. Define $\tilde{u} : \operatorname{range} 1\ (N-1) \to U$ by $\tilde{u}\ j := u_0\ j$ if $j < k$ and $u_1\ j$ else. Then $J\ P_0\ (1, N)\ x_0\ \tilde{u} \leq J\ P_0\ (1, N)\ x_0\ u_0$ by definition of the minimizer and the above equation shows the reverse inequality. The uniqueness of the minimizer implies the desired result. □

**Proposition 9.** *Let $P_0 \in L(X)$ self-adjoint and non-negative. Let $x_0 \in X$. Define $G : U \to [0, \infty)$ by*

$$G\ u := \|C\ x_0 + D\ u\|^2 + \|u\|^2 + \langle P_0(A\ x_0 + B\ u), (A\ x_0 + B\ u) \rangle. \tag{17}$$

*Then $G$ has a unique global minimum at*

$$u_m := -(Q^{-1} \circ R)\ x_0 \tag{18}$$

*and*

$$G\ u_m = \langle x_0, (C^* \circ C + A^* \circ P_0 \circ A - R^* \circ Q^{-1} \circ R)\ x_0 \rangle, \tag{19}$$

*where $R, Q$ are defined below in the proof.*

*Proof.* Let $u \in U$, then

$$\begin{aligned} G\ u &= \langle C\ x_0, C\ x_0 \rangle + \langle u, (D^* \circ D)\ u \rangle + 2 \operatorname{Re} \langle (D^* \circ C)\ x_0, u \rangle + \langle u, u \rangle \\ &\quad + \langle (P_0 \circ A)\ x_0, A x_0 \rangle + 2 \operatorname{Re} \langle (B^* \circ P_0 \circ A)\ x_0, u \rangle + \langle u, (B^* \circ P_0 \circ B)\ u \rangle \\ &= \underbrace{\langle C\ x_0, C\ x_0 \rangle + \langle (P_0 \circ A)\ x_0, A x_0 \rangle}_{=:q} \\ &\quad + 2 \operatorname{Re} \langle \underbrace{(D^* \circ C + B^* \circ P_0 \circ A)}_{=:R}\ x_0, u \rangle \\ &\quad + \langle u, \underbrace{(D^* \circ D + I + B^* \circ P_0 \circ B)}_{=:Q}\ u \rangle. \end{aligned} \tag{20}$$

Note that $Q$ is symmetric and $Q > 0$ (hence it is invertible). It is left to complete the square: For all $u, h \in U$:

$$\langle u - h, Q(u - h) \rangle = \langle u, Qu \rangle - 2 \operatorname{Re} \langle u, Qh \rangle + \langle h, Qh \rangle. \tag{21}$$

And so in particular: Let $u_m := -(Q^{-1} \circ R)x_0$ and $y_0 := q - \langle u_m, Q\ u_m \rangle$, then for all $u \in U$:

$$G\ u = \langle u - u_m, Q(u - u_m) \rangle + y_0. \tag{22}$$

Therefore $G$ has a global minimum at $u_m$ with value $y_0$. □

**Proposition 10 (Main Theorem).** *Let $P_0 \in L(X)$ self-adjoint and non-negative and $x_0 \in X$. Let $u := M\ P_0\ 1\ x_0$. Let $x := \operatorname{soln}(1, N)\ x_0\ u$. Let $P : \operatorname{range} 1\ N \to L(X)$ defined by $P\ N := P_0$ and*

$$\begin{aligned} P\ k := {} & C^* \circ C + A^* \circ P(k+1) \circ A \\ & - (C^* \circ D + A^* \circ P(k+1) \circ B) \\ & \circ (D^* \circ D + I + B^* \circ P(k+1) \circ B)^{-1} \\ & \circ (D^* \circ C + B^* \circ P(k+1) \circ A). \end{aligned} \tag{23}$$

*Then for all $k \in \operatorname{range} 1\ (N-1)$:*

$$u\ k = -(D^* \circ D + I + B^* \circ P\ (k+1) \circ B)^{-1} \circ (D^* \circ C + B^* \circ P(k+1) \circ A)\ \$\ x\ k \tag{24}$$

*and*

$$J\ P_0\ (k, N)\ (x\ k)\ f = \langle x\ k, P\ k\ (x\ k) \rangle, \tag{25}$$

*where $f : \operatorname{range} k\ (N-1) \to U,\ f\ j := u\ j$ ($f$ is the optimal input for the cost to go). In particular*

$$J\ P_0\ (1, N)\ x_0\ u = \langle x_0, P\ 1\ x_0 \rangle. \tag{26}$$

*Proof.* By Bellmans principle of optimality $u\ (N-1) = M\ P_0\ (N-1)\ (x\ (N-1))\ (N-1)$. The right hand side matches the formula in the proposition by the preceding proposition. The other assertion (for the $N-1$ case) also follows from that proposition.
Assume that the conclusion of the proposition holds for $k \in \{2, \ldots, N-1\}$. Let $v \in U$ and define $f : \operatorname{range}(k-1)\ (N-1) \to U$ by $f\ (k-1) := v$ and $f\ j := u\ j$. Then by definition and the assumption

$$J\ P_0\ (k-1, N)\ (x(k-1))\ f = \left( \|C\ (x(k-1)) + D\ v\|^2 + \|v\|^2 \right) + \langle x\ k, P\ k\ (x\ k) \rangle \tag{27}$$

and $P\ k$ is bounded, self-adjoint and non-negative. Again by Bellmans principle of optimality $u\ (k-1) = M\ P_0\ (k-1)\ (x\ (k-1))\ (k-1)$. The right hand side is given by the formula in the proposition by the preceding proposition and the above equation. The preceding proposition also concludes that $J\ P_0\ (k-1, N)\ (x(k-1))\ u = \langle x(k-1), P(k-1)(x(k-1)) \rangle$. □

# 3 Infinite Time Case

Let $X, Y, U$ Hilbert spaces and $A \in L(X), B \in L(U, X), C \in L(X, Y), D \in L(U, Y)$. In this section $\mathbb{N}$ does not contain $0$.

**Definition 11.** Define $\operatorname{soln}_\infty : X \to (\mathbb{N} \to U) \to (\mathbb{N} \to X)$ by

$$\begin{aligned} \operatorname{soln}_\infty x_0\ u\ 1 &:= x_0 \\ \operatorname{soln}_\infty x_0\ u\ (n+1) &:= A\ (\operatorname{soln}_\infty x_0\ u\ n) + B(u\ n). \end{aligned} \tag{28}$$

Define $\operatorname{out}_\infty : X \to (\mathbb{N} \to U) \to (\mathbb{N} \to Y)$ by

$$\operatorname{out}_\infty x_0\ u := C \circ (\operatorname{soln}_\infty x_0\ u) + D \circ u. \tag{29}$$

Consider the cost functional $J_\infty : X \to \ell^2(\mathbb{N}, U) \to [0, \infty]$ defined by

$$J_\infty\ x_0\ u := \sum_{n=1}^{\infty} \left( \|\operatorname{out}_\infty x_0\ u\ n\|^2 + \|u\ n\|^2 \right). \tag{30}$$

**Definition 12 (Optimizability).** The discrete time system is called optimizable if for every $x_0 \in X$ there is $u \in \ell^2(\mathbb{N}, U)$ with $J_\infty\ x_0\ u < \infty$.

**Proposition 13 (Existence and uniqueness of minimizer).** *Assume that the system is optimizable, then $J_\infty\ x_0$ has a unique minimizer for all $x_0 \in X$.*

*Proof.* The proof is almost the same as in the finite time case (only the deviation is written here). Let $H := \ell^2(\mathbb{N}, Y) \times \ell^2(\mathbb{N}, U)$. Define $V : X \to \mathcal{P}\ H$ by

$$V\ x_0 := \{(y, u) \in H : y = \mathrm{out}_\infty\, x_0\ u\}. \tag{31}$$

By the optimizability $\forall x_0 \in X : V\ x_0 \neq \varnothing$. In particular $V\ 0$ is a subspace of $H$. Furthermore $V\ x_0 = v_0 + V\ 0$ for $v_0 \in V\ x_0$ arbitrary. $V\ 0$ is closed: Let $(y, u) : \mathbb{N} \to V\ 0$ convergent to $(y_0, u_0) \in H$. Let $i \in \mathbb{N}$ and $p_i$ the $i$-th coordinate projection. Then $p_i \circ y$ converges to $p_i\ y_0$ since $\ell^2$ convergence implies pointwise convergence. Now equation 9 (adapted to the infinite time case) shows that $p_i \circ y$ converges to $p_i(\mathrm{out}_\infty\, 0\ u_0)$. Since $i$ was arbitrary this implies that $y_0 = \mathrm{out}_\infty\, 0\ u_0$ and therefore $V\ 0$ is closed. The remainder of the proof is exactly the same as in the finite case. □

**Proposition 14.** *Let $T : \mathbb{N} \to L(X)$ be an increasing sequence of symmetric, non-negative operators with*

$$\forall x \in X \exists M \in (0, \infty) : \forall n \in \mathbb{N} : \langle T\ n\ x, x\rangle \leq M. \tag{32}$$

*Then $T$ converges strongly to $T_\infty \in L_{\geq 0}(X)$ with $T_\infty \geq T\ n$ for all $n \in \mathbb{N}$.*

*Proof.* $T$ is uniformly bounded, because the condition on $T$ implies the uniform boundedness of $\sqrt{\ } \circ T$ via the uniform boundedness principle, which in turn implies the uniform boundedness of $T$ (because of the $C^*$-identity). Now for all $x_0 \in X$

$$\lim_{n\to\infty} \langle x_0, T\ n\ x_0\rangle \tag{33}$$

exists and is $\geq 0$, because the sequence is monotonically increasing and bounded from above. The polarisation identity together with the above implies, that $\forall x_0, x_1 \in X$ :

$$\lim_{n\to\infty} \langle x_0, T\ n\ x_1\rangle \tag{34}$$

exists. Therefore $s : X \times X \to \mathbb{K}$, $s\ x_0\ x_1 := \lim_{n\to\infty}\langle x_0, T\ n\ x_1\rangle$ is a well defined, conjugate symmetric, positive semi-definite, sesquilinear form. Furthermore for all $x_0, x_1 \in X$:

$$|s\ x_0\ x_1| = \lim_{n\to\infty} |\langle x_0, T\ n\ x_1\rangle| \leq \sup_{n\in\mathbb{N}} \|T\ n\| \cdot \|x_0\| \cdot \|x_1\|. \tag{35}$$

Therefore $s$ is continuous and so there exists a symmetric and non-negative operator $T_\infty \in L(X)$ with for all $x_0, x_1 \in X$:

$$s\ x_0\ x_1 = \langle x_0, T_\infty\ x_1\rangle. \tag{36}$$

Evidently $T_\infty \geq T\ n$ for all $n \in \mathbb{N}$. Define $S : \mathbb{N} \to L_{\geq 0}(X)$ by $S\ n := T_\infty - T\ n$. Then $S \to 0$ in the weak operator topology. For all $n \in \mathbb{N}$ and $x \in X$:

$$\|S\ n\ x\|^2 \leq \sup_{m\in\mathbb{N}} \|S\ m\| \cdot \langle S\ n\ x, x\rangle. \tag{37}$$

Which implies that $T \to T_\infty$ strongly (the inequality in the above equation follows from the Cauchy-Schwarz inequality and using the $\sqrt{\ }$). □

**Definition 15.** Define $T : \mathbb{N} \to L(X)$ as follows: $T\ 1 := 0$ and

$$\begin{aligned} T\ N := {} & C^* \circ C + A^* \circ T(N-1) \circ A \\ & - (C^* \circ D + A^* \circ T(N-1) \circ B) \\ & \circ (D^* \circ D + I + B^* \circ T(N-1) \circ B)^{-1} \\ & \circ (D^* \circ C + B^* \circ T(N-1) \circ A). \end{aligned} \tag{38}$$

**Proposition 16.** *For all $N \in \mathbb{N}$:*

1. $\forall x_0 \in X : \langle x_0, T\ N\ x_0\rangle = \inf_{u\in\ell^2(\mathrm{range}\,1\ (N-1), U)} J\ 0\ (1, N)\ x_0\ u.$
2. $0 \leq T\ N \leq T\ (N+1).$

*Proof.* To the first point: Let $N \in \mathbb{N}$. Define $P : \mathrm{range}\,1\ N \to L(X)$ by $P\ j := T\ (N + 1 - j)$, then evidently $P\ N = 0$ and $P\ 1 = T\ N$. Furthermore by definition for all $j \in \mathrm{range}\,1\ (N - 1)$

$$\begin{aligned} P\ j &= C^* \circ C + A^* \circ P(j+1) \circ A \\ &\quad - (C^* \circ D + A^* \circ P(j+1) \circ B) \\ &\quad \circ (D^* \circ D + I + B^* \circ P(j+1) \circ B)^{-1} \\ &\quad \circ (D^* \circ C + B^* \circ P(j+1) \circ A). \end{aligned} \tag{39}$$

The main theorem of the last section concludes. The second point is a consequence of the first. □

**Proposition 17 (Main theorem).** *Assume that the discrete time system is optimizable. Then*

1. *$T$ converges strongly to $\Pi \in L_{\geq 0}(X)$ with $\Pi \geq T\ n$ for all $n \in \mathbb{N}$.*
2. *$\Pi$ satisfies the algebraic riccati equation (ARE):*

$$\begin{aligned} \Pi &= C^* \circ C + A^* \circ \Pi \circ A \\ &\quad - (C^* \circ D + A^* \circ \Pi \circ B) \\ &\quad \circ (D^* \circ D + I + B^* \circ \Pi \circ B)^{-1} \\ &\quad \circ (D^* \circ C + B^* \circ \Pi \circ A). \end{aligned} \tag{40}$$

3. *Any other symmetric non-negative solution $\Pi_2$ to the ARE satisfies $\Pi \leq \Pi_2$.*
4. *For all $x_0 \in X$: Define $\tilde{u} : \mathbb{N} \to U$ by*

$$\tilde{u}\ k := -(D^* \circ D + I + B^* \circ \Pi \circ B)^{-1} \circ (D^* \circ C + B^* \circ \Pi \circ A)\ \$\ (\mathrm{soln}_\infty\ x_0\ \tilde{u}\ k). \tag{41}$$

*Then*

$$\inf_{u \in \ell^2} J_\infty\ x_0\ u = J_\infty\ x_0\ \tilde{u} = \langle x_0, \Pi\ x_0 \rangle. \tag{42}$$

*Proof.* The first assertion follows from the optimizability and the preceding two propositions. That $\Pi$ solves the ARE follows from the definition of $T$ and the strong convergence.
To "4.": Let $x_0 \in X$. For all $N \in \mathbb{N}$ :

$$\langle x_0, T\ N\ x_0 \rangle = \inf_{u \in \ell^2} J\ 0\ (1, N)\ x_0\ u \leq \inf_{u \in \ell^2} J_\infty\ x_0\ u. \tag{43}$$

This implies that $\langle x_0, \Pi\ x_0 \rangle \leq \inf_{u \in \ell^2} J_\infty\ x_0\ u$. On the other hand

$$\inf_{u \in \ell^2} J_\infty\ x_0\ u \leq J_\infty\ x_0\ \tilde{u} = \lim_{N \to \infty} J\ 0\ (1, N)\ x_0\ \tilde{u}|_{\ldots} \leq \lim_{N \to \infty} J\ \Pi\ (1, N)\ x_0\ \tilde{u}|_{\ldots} = \langle x_0, \Pi\ x_0 \rangle. \tag{44}$$

Where the second inequality is due to the fact that $\Pi$ is non-negative and the last equality due to the fact that $\Pi$ solves the ARE together with the main theorem of the preceding section. The fact that $\tilde{u} \in \ell^2$ follows from $\sum_{n=1}^{\infty} \|\tilde{u}\ n\|^2 \leq \lim_{N \to \infty} J\ \Pi\ (1, N)\ x_0\ \tilde{u}|_{\ldots} < \infty$.
To "3.": Let $x_0 \in X$ and define $\tilde{u}$ as in point "4." of the proposition, but with $\Pi_2$ in place of $\Pi$. Then

$$\begin{aligned} \langle x_0, \Pi\ x_0 \rangle &= \inf_{u \in \ell^2} J_\infty\ x_0\ u \leq J_\infty\ x_0\ \tilde{u} = \lim_{N \to \infty} J\ 0\ (1, N)\ x_0\ \tilde{u}|_{\ldots} \\ &\leq \lim_{N \to \infty} J\ \Pi_2\ (1, N)\ x_0\ \tilde{u}|_{\ldots} = \langle x_0, \Pi_2\ x_0 \rangle. \end{aligned} \tag{45}$$

Therefore $\Pi \leq \Pi_2$. □

**Proposition 18.** *Let $\Pi_2 \in L_{\geq 0}(X)$ be a solution to the ARE. Then the system is optimizable.*

*Proof.* Let $x_0 \in X$ and define $\tilde{u}$ as in point "4." of the preceding proposition, but with $\Pi_2$ in place of $\Pi$. Then

$$\begin{aligned} J_\infty\ x_0\ \tilde{u} &= \lim_{N \to \infty} J\ 0\ (1, N)\ x_0\ \tilde{u}|_{\ldots} \\ &\leq \lim_{N \to \infty} J\ \Pi_2\ (1, N)\ x_0\ \tilde{u}|_{\ldots} = \langle x_0, \Pi_2\ x_0 \rangle < \infty. \end{aligned} \tag{46}$$

□