Control and Observation Operators

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Contents

The talk is based on the results from section 4.1 titled "Solutions of non-homogeneous differential equations" of [TW09].

1 The Inhomogeneous Initial Value Problem (iIVP)

Throughout this section let:

1. X (state space) and U (input space) Banach spaces,
2. T a C_0-semigroup on X with generator A,
3. $B \in L(U, X)$.

Definition 1.1 (Types of Solutions). A function $x : [0, \infty) \to X$ is called a

- **Classical Solution** of the iIVP associated to (A, B) with initial value $x_0 \in \mathrm{dom}(A)$ and input $u \in C([0, \infty), U)$ if:
  1. $x \in C^1([0, \infty), X)$,
  2. $x(0) = x_0$,
  3. $\forall t \in [0, \infty) : x(t) \in \mathrm{dom}(A)$ and
  $$\dot{x}(t) = Ax(t) + Bu(t).$$
- **Strong Solution**[1] of the iIVP associated to (A, B) with initial value $x_0 \in X$ and input $u \in L^1_{\mathrm{loc}}([0, \infty), U)$ if:
  1. $x \in C([0, \infty), X)$,
  2. $x \in L^1_{\mathrm{loc}}([0, \infty), Y)$, where $Y := (\mathrm{dom}(A), \|\cdot\|_{\mathrm{gr}(A)})$,

[1] In [TW09] strong solutions are simply called solutions (in X).

3. $\forall t \in [0,\infty)$:

$$x(t) = x_0 + \int_0^t Ax(s) + Bu(s)ds.$$

- **Mild Solution** of the iIVP associated to (A,B) with initial value $x_0 \in X$ and input $u \in L^1_{\text{loc}}([0,\infty),U)$ if:

  1. $x \in C([0,\infty),X)$,
  2. $\forall t \in [0,\infty)$: $\int_0^t x(s)ds \in \text{dom}(A)$ and

$$x\ t = x_0 + A\int_0^t x(s)ds + \int_0^t Bu(s)ds.$$

Proposition 1.2.

classical solution $\Rightarrow$ strong solution $\Rightarrow$ mild solution.

Lemma 1.3. *Define $\Phi : [0,\infty) \to L(L^1_{\text{loc}}([0,\infty),U),X)$ by*

$$\Phi(t)u := \int_0^t T(t-s)Bu(s)ds. \tag{1.1}$$

Then Φ is well-defined and strongly continuous.

Proof. To show well-definedness: Let $u \in L^1_{\text{loc}}([0,\infty)$ and $\in [0,\infty)$. Then

$$\| \int_0^t T(t-s)Bu(s)ds\| \leq \sup_{s\in[0,t]} \|T(s)\| \cdot \int_0^t \|u(s)\|ds.$$

To show strong continuity: Let $t \in [0,\infty)$ and $\delta > 0$. Then

$$\begin{aligned}\Phi(t+\delta)u - \Phi(t)u &= \int_0^{t+\delta} T(t+\delta-s)Bu(s)ds - \int_0^t T(t-s)Bu(s)ds\\ &= \int_0^t (T(t+\delta-s) - T(t-s))Bu(s)ds + \int_t^{t+\delta} T(t+\delta-s)Bu(s)ds\\ &= (T(\delta)-I)\Phi(t)u + \int_t^{t+\delta} T(t+\delta-s)Bu(s)ds.\end{aligned}$$

The norm of the first summand can be made small since T is strongly continuous and the norm of the second summand by the dominated convergence theorem. On the other hand if $t-\delta \geq 0$, then

$$\begin{aligned}\Phi(t-\delta)u - \Phi(t)u &= \int_0^{t-\delta} T(t-\delta-s)Bu(s)ds - \int_0^t T(t-s)Bu(s)ds\\ &= \int_0^{t-\delta} (T(t-\delta-s) - T(t-s))Bu(s)ds - \int_{t-\delta}^t T(t-s)Bu(s)ds\end{aligned}$$

and both summands can be seen to converge to 0 as $\delta \to 0$ by Lebesgues theorem of dominated convergence. □

Theorem 1.4 (Existence and Uniqueness of Mild Solutions: the Principle of Duhamel). *Define $x : [0,\infty) \times X \times L^1_{\text{loc}}([0,\infty),U) \to X$ by*

$$x(t,x_0,u) := T(t)x_0 + \underbrace{\int_0^t T(t-s)Bu(s)ds}_{=\Phi(t)u}. \tag{1.2}$$

Let $x_0 \in X$ and $u \in L^1_{\text{loc}}([0,\infty),U)$. Then $x(\cdot,x_0,u)$ is the unique mild solution to the iIVP associated to (A,B) with input u and initial value x_0.

Proof. Let $x := x(\cdot, x_0, u)$. The continuity of x follows from the fact that Φ and T are strongly continuous. Let $t \in [0, \infty)$: Then

$$\int_0^t x(s)ds = \int_0^t T(s)x_0 ds + \int_0^t \int_0^s T(s-\sigma)Bu(\sigma)d\sigma ds.$$

The first summand is in the domain of A and $A(\int_0^t T(s)x_0 ds) = T(t)x_0 - x_0$ by a well known result. For the second summand: Let

$$S := \{(s, \sigma) \in [0, t]^2 : 0 \leq s \leq t, 0 \leq \sigma \leq s\}.$$

Then, using Fubinis-Theorem and the substitution $\tau \mapsto \tau - \sigma$

$$\begin{aligned}
\int_0^t \int_0^s T(s-\sigma)Bu(\sigma)d\sigma ds &= \int_{[0,t]^2} \chi_S(s,\sigma)T(s-\sigma)Bu(\sigma)d(\sigma, s) \\
&= \int_0^t \int_0^t \chi_S(s,\sigma)T(s-\sigma)\ Bu(\sigma)dsd\sigma \\
&= \int_0^t \int_\sigma^t T(s-\sigma)Bu(\sigma)dsd\sigma \\
&= \int_0^t \int_0^{t-\sigma} T(s)Bu(\sigma)dsd\sigma
\end{aligned}$$

Now for all $\sigma \in [0, t] : \int_0^{t-\sigma} T(s)Bu(\sigma)ds \in \mathrm{dom}(A)$ and

$$A\Big(\int_0^{t-\sigma} T(s)Bu(\sigma)ds\Big) = T(t-\sigma)Bu(\sigma) - Bu(\sigma)$$

by the same well known result. Since $\int_0^t T\ (t-\sigma)Bu(\sigma) - Bu(\sigma)d\sigma$ exists and A is closed (and the well known property of the Bochner integral) the above implies, that $\int_0^t \int_0^s T(s-\sigma)Bu(\sigma)d\sigma ds \in \mathrm{dom}(A)$ and

$$A\Big(\int_0^t \int_0^s T(s-\sigma)\ (f\ \sigma)d\sigma ds\Big) = \int_0^t T(t-\sigma)Bu(\sigma) - Bu(\sigma)d\sigma.$$

Putting everything together: $\int_0^t x\ sds \in \mathrm{dom}(A)$ (since it is a vector space) and

$$A\Big(\int_0^t x(s)ds\Big) = T(t)x_0 - x_0 + \int_0^t T(t-\sigma)Bu(\sigma)d\sigma - \int_0^t Bu(\sigma)d\sigma.$$

Therefore x is a mild solution. Let y be another mild solution with input u and initial value x_0. Let $z := y - x$. Then $z \in C([0, \infty), X)$ and for all $t \in [0, \infty)$: $\int_0^t z(s)ds \in \mathrm{dom}(A)$ and

$$\begin{aligned}
z(t) &= y(t) - x(t) \\
&= A\Big(\int_0^t z(s)ds\Big).
\end{aligned}$$

Let $t \in (0, \infty)$ and define $g : [0, t] \to X$ by

$$g(s) := T(t-s)\Big(\int_0^s z(\sigma)d\sigma\Big).$$

Then g is differentiable and for all $s \in [0, t]$:

$$g'\ s = T(t-s)z(s) - T(t-s)\underbrace{A\Big(\int_0^s z(\sigma)d\sigma\Big)}_{=z(s)} = 0.$$

Therefore g is constant and so

$$0 = g(0) = g(t) = \int_0^t z(\sigma)d\sigma.$$

Since t was arbitrary it follows that $z = 0$ by the continuity of z. □

2 The Unilateral Left Shift Semigroup

Definition 2.1 (Core). Let X be a Banach space and $A : X \supset \mathrm{dom}(A) \to X$ a closed operator. A subspace $Y \subset \mathrm{dom}(A)$ is called a *core* of A if $\overline{A|_Y} = A$.

Proposition 2.2. *Let T be a C_0-semigroup with generator A on the Banach space X. Let $Y \subset \mathrm{dom}(A)$ a subspace that is dense in X and T invariant. Then Y is a core of A.*

Proof. See Proposition 1.7 of [EN99]. □

Proposition 2.3 (Almost Everywhere Pointwise Evaluation of L^p-valued Integrals). *Let*

1. *X a Banach space,*
2. *$p \in [1, \infty)$,*
3. *$(S, \mathscr{A}, \mu), (T, \mathscr{B}, \nu)$ σ-finite measure spaces,*
4. *$F : S \to L^p(T, X)$ Bochner integrable.*

Then there exists a $(\mu \times \nu)$-measurable function $g : S \times T \to X$ with the following properties:

1. *for μ-almost all $s \in S : [T \ni t \mapsto g(s, t)] = F(s)$,*
2. *for ν-almost all $t \in T$: $S \ni s \mapsto g(s, t)$ is Bochner integrable and*

$$\Big(\int_S F(s) d\mu(s) \Big)(t) = \int_S g(s, t) d\mu(s), \tag{2.1}$$

3. *g is unique in the sense that if $h : S \times T \to X$ is measurable and satisfies 1., then $h = g$ $(\mu \times \nu)$-almost everywhere.*

Proof. See Proposition 1.2.25 in [Hyt+16]. □

Proposition 2.4 (Unilateral Left-Shift Semi-Group). *Let X be a Banach space and $p \in [1, \infty)$. Define the* unilateral left-shift semi-group *S by*

$$\begin{aligned} S : [0, \infty) &\longrightarrow L(L^p([0, \infty), X)) \\ t &\longmapsto f \mapsto \big(s \mapsto f(s + t)\big). \end{aligned} \tag{2.2}$$

Then

1. *S is a C_0-semigroup,*
2. *the generator D of S is the closure of*

$$\begin{aligned} D_0 : L^p([0, \infty), X)) \supset C_c^\infty([0, \infty), X) &\longrightarrow L^p([0, \infty), X)) \\ f &\longmapsto f' \end{aligned} \tag{2.3}$$

3. *the Resolvent R of D satisfies $\forall \lambda \in \mathbb{C}$ with $\mathrm{Re}\, \lambda > \omega_0(S)$:*

$$\text{for almost all } s \in [0, \infty) : (R(\lambda) f)(s) = \int_s^\infty \exp(-\lambda(\tau - s)) f(\tau) d\tau. \tag{2.4}$$

In particular this shows, that every element of $\mathrm{dom}(D)$ has a (unique) continuous representant.

Proof. To "1.": S is C_0-semi-group: skipped.
To "2.": The space $C_c^\infty([0, \infty), X)$ is a dense, S invariant subspace. If we can show that $C_c^\infty([0, \infty), X) \subset \mathrm{dom}(D)$ and that A is given by differentiation on this space, then we are finished by proposition 2.2. However this is a simple consequence of the FTC and the compact support property: Let $f \in C_c^\infty([0, \infty), X)$ and $b \in [0, \infty)$ such that the support of f is contained in $[0, b]$. For all $s \in [0, \infty), h \in (0, \infty)$:

$$\frac{S\ h\ f\ s - f\ s}{h} - f'(s) = \frac{f\ (s + h) - f\ s}{h} - f'(s) = \frac{1}{h} \cdot \int_s^{s+h} f'(t) - f'(s) dt$$

Since f' is continuous and supported in the compact set $[0,b]$ it is uniformly continuous. Let $\varepsilon > 0$. Therefore (by definition) there exists $\delta \in (0,\infty)$:

$$\forall x, y \in [0,\infty) : |x-y| < \delta \Rightarrow \|f'(x) - f'(y)\| < \varepsilon$$

Then for all $h \in (0,\delta)$ (note that the support of $S\ h\ f$ is contained in $[0,b]$):

$$\|\frac{S\ h\ f - f}{h} - f'\|_{L^1} \leq b \sup_{s\in[0,b]} \|\frac{S\ h\ f\ s - f\ s}{h} - f'(s)\| \leq \sup_{s\in[0,b]} \frac{1}{h}\int_s^{s+h} \underbrace{\|f'(t) - f'(s)\|}_{<\varepsilon} dt < b \cdot \varepsilon.$$

To "3.": Let $f \in L^p([0,\infty), X)$ and $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda > \omega_0(S)$. Define $F : [0,\infty) \to L^p([0,\infty), X)$ by

$$F(t) := \exp(-\lambda \cdot t) \cdot S(t)\ f$$

Then

$$R(\lambda) f = \int_0^\infty F(t) dt.$$

The function $g : [0,\infty) \times [0,\infty) \to X$ defined by

$$g(s,t) := \exp(-\lambda t) f(t+s)$$

is product measurable and satisfies $s \mapsto g(s,t) = F(t)$ for almost all $t \in [0,\infty)$. Therefore using proposition 2.3 for almost all $s \in [0,\infty)$:

$$\begin{aligned}(R(\lambda)\ f)(s) &= \left(\int_0^\infty F(t)dt\right)(s)\\ &= \int_0^\infty \exp(-\lambda t) f(s+t) dt\\ &= \int_s^\infty \exp(-\lambda(t-s)) f(t) dt\end{aligned}$$

□

Definition 2.5 (Sobolev Spaces). In the situation of proposition 2.4: Define

$$W^{1,p}([0,\infty), X) := (\operatorname{dom} D, \|\cdot\|_{1,p}),$$

where

$$\|f\|_{1,p} := \left(\|f\|_{L^p}^p + \|Df\|_{L^p}^p\right)^{1/p}.$$

Then $W^{1,p}([0,\infty), X)$ is a Banach space, because the norm $\|\cdot\|_{1,p}$ is (equivalent to) the graph norm of D and D is closed. Furthermore we define

$$W^{1,p}_{\mathrm{loc}}([0,\infty), X) := \{f \in L^p_{\mathrm{loc}}([0,\infty), X) : \forall t \in (0,\infty) \exists g \in W^{1,p}([0,\infty), X) \text{ with } g|_{[0,t]} = f|_{[0,t]}\}.$$

If X is a Hilbert space we also define $H^1([0,\infty), X) := W^{1,2}([0,\infty), X)$ and $H^1_{\mathrm{loc}}([0,\infty), X) := W^{1,2}_{\mathrm{loc}}([0,\infty), X)$. In this case H^1 is a Hilbert space as well.

3 Existence of Classical Solutions

Throughout this section let:

1. X (state space) and U (input space) Banach spaces,
2. T a C_0-semigroup on X with generator A,
3. $B \in L(U, X)$,
4. $p \in [1,\infty)$,
5. $\delta_0 : W^{1,p}([0,\infty), U) \to U$ the point evaluation of the unique continuous representant at zero,

6. S the unilateral left shift semigroup on $L^p([0,\infty),U)$ and D its generator,

7. $\Phi : [0,\infty) \to L(L^p([0,\infty),U),X)$ defined by

$$\Phi(t)u := \int_0^t T(t-s)Bu(s)ds. \tag{3.1}$$

Theorem 3.1. *Let $\mathcal{X} := X \times L^p([0,\infty),U)$. Define $\mathcal{T} : [0,\infty) \to L(\mathcal{X})$ by*

$$\mathcal{T}(t) := \begin{pmatrix} T(t) & \Phi(t) \\ 0 & S(t) \end{pmatrix}. \tag{3.2}$$

Then:

1. *$\mathcal{T}$ is a C_0-semigroup,*
2. *the generator $\mathcal{A}$ of $\mathcal{T}$ is given by*
$$\mathcal{A} = \begin{pmatrix} A & B\delta_0 \\ 0 & D \end{pmatrix} \tag{3.3}$$
with
$$\operatorname{dom}(\mathcal{A}) = \operatorname{dom}(A) \times W^{1,p}([0,\infty),U), \tag{3.4}$$
3. *for all $(x_0,u) \in \mathcal{X}$ and $\lambda \in \mathbb{C}$ in some right half plane:*
$$\int_0^\infty \exp(-\lambda t)x(t,x_0,u)dt = (\lambda I - A)^{-1}\Big(x_0 + \int_0^\infty \exp(-\lambda t)Bu(t)dt\Big). \tag{3.5}$$

Proof. To "1.": Clearly $\mathcal{T}(0) = I$. To show the functional equation let $t,s \in [0,\infty)$. Then

$$\mathcal{T}(s)\mathcal{T}(t) = \begin{pmatrix} T(s) & \Phi(s) \\ 0 & S(s) \end{pmatrix}\begin{pmatrix} T(t) & \Phi(t) \\ 0 & S(t) \end{pmatrix} = \begin{pmatrix} T(t+s) & T(s)\Phi(t) + \Phi(s)S(t) \\ 0 & S(t+s) \end{pmatrix}.$$

Therefore it is left to show that

$$\Phi(t+s) = T(s)\Phi(t) + \Phi(s)S(t).$$

To this end let $u \in L^p([0,\infty),U)$. Then

$$\begin{aligned} T(s)\Phi(t)u + \Phi(s)S(t)u &= T(s)\int_0^t T(t-\sigma)Bu(\sigma)d\sigma + \int_0^s T(s-\sigma)B(S(t)u)(\sigma)d\sigma \\ &= \int_0^t T(t+s-\sigma)Bu(\sigma)d\sigma + \int_0^s T(s-\sigma)Bu(t+\sigma)d\sigma \\ &= \int_0^t T(t+s-\sigma)Bu(\sigma)d\sigma + \int_t^{t+s} T(t+s-\sigma)Bu(\sigma)d\sigma \\ &= \Phi(t+s)f. \end{aligned}$$

The strong continuity follows from the fact that S, T, Φ are strongly continuous.
To "2.": Let $R_{\mathcal{A}}, R_A, R_D$ be the Resolvent of $\mathcal{A}, A, D$. Let $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda$ larger than $\omega_0(\mathcal{T}), \omega_0(S)$ and $\omega_0(T)$. Let $u \in L^p([0,\infty),U)$. Then

$$R_{\mathcal{A}}(\lambda)\begin{pmatrix} x_0 \\ u \end{pmatrix} = \int_0^\infty \exp(-\lambda t)\mathcal{T}(t)\begin{pmatrix} x_0 \\ u \end{pmatrix}dt = \begin{pmatrix} R_A(\lambda)x_0 + \int_0^\infty \exp(-\lambda t)\Phi(t)u dt \\ R_D(\lambda)u \end{pmatrix}.$$

Now let

$$M := \{(t,s) \in [0,\infty)^2 : s \le t\}.$$

$$\begin{aligned}\int_0^\infty \exp(-\lambda t)\Phi(t)udt &= \int_0^\infty \exp(-\lambda t)\int_0^t T(t-s)Bu(s)dsdt\\ &= \int_{[0,\infty)^2}\chi_M(s,t)\exp(-\lambda t)T(t-s)Bu(s)d(s,t)\\ &= \int_0^\infty\int_s^\infty \exp(-\lambda t)T(t-s)Bu(s)dtds\\ &= \int_0^\infty\int_0^\infty \exp(-\lambda t)T(t)\exp(-\lambda s)Bu(s)dtds\\ &= \int_0^\infty \exp(-\lambda t)T(t)\int_0^\infty \exp(-\lambda s)Bu(s)dsdt\\ &= R_A(\lambda)\int_0^\infty \exp(-\lambda s)Bu(s)ds.\end{aligned}$$

On the other hand

$$\left(\lambda I - \begin{pmatrix} A & B\delta_0 \\ 0 & D\end{pmatrix}\right)^{-1} = \begin{pmatrix} R_A(\lambda) & R_A(\lambda)B\delta_0 R_D(\lambda) \\ 0 & R_D(\lambda)\end{pmatrix},$$

because

$$\begin{pmatrix} \lambda I - A & -B\delta_0 \\ 0 & \lambda I - D\end{pmatrix}\begin{pmatrix} R_A(\lambda) & R_A(\lambda)B\delta_0 R_D(\lambda) \\ 0 & R_D(\lambda)\end{pmatrix} = \begin{pmatrix} I & (\lambda I - A)R_A(\lambda)B\delta_0R_D(\lambda) - B\delta_0R_D(\lambda) \\ 0 & (\lambda I - D)R_D(\lambda)\end{pmatrix} = I.$$

Other equation analogue. Now let $u \in L^p([0,\infty), U)$. Then

$$R_A(\lambda)B\delta_0R_D(\lambda)u = R_A(\lambda)B\int_0^\infty \exp(-\lambda\tau)u(\tau)d\tau = R_A(\lambda)\int_0^\infty \exp(\lambda\tau)Bu(\tau)d\tau,$$

because for almost all $s \in [0,\infty) : (R_D(\lambda)u)(s) = \int_s^\infty \exp(-\lambda(\tau - s))f(\tau)d\tau$. This shows that

$$\int_0^\infty \exp(-\lambda t)\Phi(t)udt = R_A(\lambda)B\delta_0R_D(\lambda)u$$

and so in total

$$(\lambda \cdot I - \mathcal{A})^{-1} = \left(\lambda I - \begin{pmatrix} A & B\delta_0 \\ 0 & D\end{pmatrix}\right)^{-1}.$$

□

Corollary 3.2. *Let $x_0 \in \operatorname{dom}(A)$ and $u \in W^{1,p}([0,\infty), U)$. Then there exists a classical solution to the iIVP associated to (A,B) with initial value x_0 and input u.*

Proof. By assumption $(x_0, u) \in \operatorname{dom}(\mathcal{A})$. Therefore $x : [0,\infty) \to \mathcal{X}$ defined by

$$x(t) := \begin{pmatrix} x_1(t) \\ x_2(t)\end{pmatrix} := \mathcal{T}(t)\begin{pmatrix} x_0 \\ u\end{pmatrix}$$

is in $C^1([0,\infty),\mathcal{X})$ and satisfies $x(0) = (x_0, u)$. Furthermore for all $t \in [0,\infty)$:

$$\begin{pmatrix} \dot{x}_1(t) \\ \dot{x}_2(t)\end{pmatrix} = \dot{x}(t) = \mathcal{A}x(t) = \begin{pmatrix} A & B\delta_0 \\ 0 & D\end{pmatrix}\begin{pmatrix} x_1(t) \\ x_2(t)\end{pmatrix} = \begin{pmatrix} Ax_1(t) + Bx_2(0) \\ Dx_2(t)\end{pmatrix}.$$

Now $Bx_2(0) = B(S(t)u)(0) = Bu(t)$. This implies, that x_1 is the classical solution to the iIVP associated to (A,B) with initial value x_0 and input u. □

Corollary 3.3. *Let $x_0 \in \operatorname{dom}(A)$ and $u \in W^{1,p}_{\mathrm{loc}}([0,\infty), U)$. Then there exists a classical solution to the iIVP associated to (A,B) with initial value x_0 and input u.*

Proof. Let $\tau \in (0,\infty)$ and $g \in W^{1,p}([0,\infty), U)$ with $g|_{[0,\tau]} = u|_{[0,\tau]}$. Then for all $t \in [0,\tau]$

$$x(t, x_0, u) = x(t, x_0, g).$$

And $x(\cdot, x_0, g)$ is a classical solution by the preceding. □

Throughout let:

1. $(X, \|\cdot\|)$ a Banach space,
2. T a C_0-semigroup on X with generator A,
3. $\beta \in \rho(A)$.

4 Interpolation-, Extrapolation-Spaces and Semigroups

4.1 The Interpolated Semigroup

Definition 4.1 (Interpolated Space). Define the *interpolated space* $(X_1, \|\cdot\|_1)$ by

$$X_1 := \mathrm{dom}(A)$$

and

$$\|x\|_1 := \|(\beta I - A)x\|.$$

Proposition 4.2. *The following are true:*

1. $(\beta I - A)$ *is a surjective isometry* $X_1 \to X$
2. $\|\cdot\|_1$ *is equivalent to the graph norm of* A *(and so* X_1 *is a Banach space and* $\|\cdot\|_1$ *is stronger than* $\|\cdot\|$*)*

Proposition 4.3 (Interpolated Semigroup). *Define the* interpolated semigroup[2] $T_1 : [0, \infty) \to L(X_1)$ *by*

$$T_1(t) := (\beta I - A)^{-1} T(t)(\beta I - A).$$

Then

1. T_1 *is a* C_0*-semigroup,*
2. *The generator* A_1 *of* T_1 *is given by the restriction of* A *to* $\mathrm{dom}(A^2)$*,*
3. T_1 *is the restriction of* T*.*

4.2 The Extrapolated Semigroup

Definition 4.4 (Extrapolated Space). Define the *extrapolated space* $(X_{-1}, \|\cdot\|_{-1})$ as the completion of X with respect to the norm $\|\cdot\| \circ (\beta \cdot I - A)^{-1}$.

Proposition 4.5. *The following is true:*

1. $\beta I - A$ *(resp.* A*) is an isometry with dense range (resp. continuous) as an operator* $(\mathrm{dom}\, A, \|\cdot\|_0) \to X_{-1}$*,*
2. $\beta I - A_{-1}$ *is the unique extension of* $\beta I - A$ *to a surjective isometry* $X \to X_{-1}$*, where* $A_{-1} \in L(X, X_{-1})$ *is the unique continuous extension of* A*.*

Proposition 4.6 (Extrapolated Semigroup). *Define the* extrapolated semigroup[3] $T_{-1} : [0, \infty) \to L(X_{-1})$ *by*

$$T_{-1}(t) := (\beta I - A_{-1}) T(t)(\beta I - A_{-1})^{-1}.$$

Then

1. T_{-1} *is a* C_0*-semigroup,*
2. *The generator of* T_{-1} *is* A_{-1}*,*
3. T_{-1} *extends* T*.*

Proof. The first two points are obvious since T_{-1} is similar to T and since $(\beta I - A_{-1})(\mathrm{dom}(A)) = X$. And for $x_0 \in X$:

$$\begin{aligned}(\beta I - A_{-1})A(\beta I - A_{-1})^{-1}x &= (\beta I - A_{-1})A(\beta I - A)^{-1}x \\ &= (\beta I - A_{-1})(\beta(\beta I - A)^{-1}x - x) \\ &= \beta I - \beta I + A_{-1}x.\end{aligned}$$

since

$$I = \beta(\beta I - A)^{-1} - A(\beta I - A)^{-1}.$$

The third point follows from the fact, that T commutes with its generator and that $\|\cdot\|$ is stronger than $\|\cdot\|_{-1}$. □

[2] in [TW09] T_1 is denoted by the same symbol as the original semigroup

[3] in [TW09] T_{-1} is denoted by the same symbol as the original semigroup

Diagram 4.7 (Inter-/Extra-Polation Summary). *The relationship between the inter-/extra-polation spaces and semigroups are visualised in the following commutative diagram, where both squares commute for any $t \in [0, \infty)$ and all the vertical arrows are surjective isometries:*

$$\begin{array}{ccc}
X_{-1} & \xrightarrow{T_{-1}(t)} & X_{-1} \\
{\scriptstyle \beta I - A_{-1}} \big\uparrow & & \big\downarrow {\scriptstyle (\beta I - A_{-1})^{-1}} \\
X & \xrightarrow{T(t)} & X \\
{\scriptstyle \beta I - A} \big\uparrow & & \big\downarrow {\scriptstyle (\beta I - A)^{-1}} \\
X_1 & \xrightarrow{T_1(t)} & X_1
\end{array}$$

4.3 The Hilbert Space Case

Assume (only for this subsection) that X is a Hilbert space. Let:

1. $J_X : X \to X'$ the surjective and anti-linear Riesz isometry,
2. T^* the adjoint semigroup of T (whose generator is A^*),
3. X_1^d the interpolation space associated to T^* and $\bar{\beta}$ (possible since $\bar{\beta} \in \rho(A^*)$),
4. $i_d : X_1^d \to X$ the natural injection.

Proposition 4.8 (Summary of First Seminar). *The following is true:*

1. *i_d is continuous and has dense range,*
2. *$(i_d)' \circ J_X : X \to (X_1^d)'$ has dense range,*
3. *for all $x \in X$: $\|x\|_{-1} = \|i_d'(J_X(x))\|$,*
4. *$(i_d)' \circ J_X$ extends to a unique anti-linear and surjective isometry $J : X_{-1} \to (X_1^d)'$,*
5. *For all $t \in [0, \infty)$:*
$$T_{-1} = J^{-1} \circ ((T^*)_1(t))' \circ J.$$

Proof. This was shown in the first seminar. □

Diagram 4.9 (Inter-/Extra-Polation Summary, Hilbert Case). *The relationship between the inter-/extra-polation spaces and semigroups in the Hilbert space case are visualised in the following commutative diagram, where all three squares commute for any $t \in [0, \infty)$ and all the vertical arrows are surjective isometries:*

$$\begin{array}{ccc}
(X_1^d)' & \xrightarrow{((T^*)_1(t))'} & (X_1^d)' \\
{\scriptstyle J} \big\uparrow & & \big\downarrow {\scriptstyle J^{-1}} \\
X_{-1} & \xrightarrow{T_{-1}(t)} & X_{-1} \\
{\scriptstyle \beta I - A_{-1}} \big\uparrow & & \big\downarrow {\scriptstyle (\beta I - A_{-1})^{-1}} \\
X & \xrightarrow{T(t)} & X \\
{\scriptstyle \beta I - A} \big\uparrow & & \big\downarrow {\scriptstyle (\beta I - A)^{-1}} \\
X_1 & \xrightarrow{T_1(t)} & X_1
\end{array}$$

5 Admissible Control Operators

This section is based on section 4.2 of [TW09]. In this section let:

1. U a Banach space called the *input space*,
2. $p \in [1, \infty)$,
3. $B \in L(U, X_{-1})$ called the *control operator*,
4. S_l (resp. S_r) the unilateral left (resp. right) shift semigroup on $L^p([0,\infty), U)$.

Definition 5.1 (Truncation Operator). Define the *truncation operator*

$$P : [0,\infty) \to L\big(L^p_{\mathrm{loc}}([0,\infty), U), L^p([0,\infty), U)\big)$$

by

$$(P(t)u)(s) := \begin{cases} u(s), & \text{if } s \le t, \\ 0, & \text{else.} \end{cases}$$

Definition 5.2 (Controllability Map). Define the *controllability map* $\Phi : [0,\infty) \to L(L^p([0,\infty), U), X_{-1})$ by

$$\Phi(t) := \int_0^t T_{-1}(t-s)Bu(s)ds.$$

Proposition 5.3 (Causality Property). *For all $s, t \in [0,\infty)$ with $s \ge t$ and $u \in L^p_{\mathrm{loc}}([0,\infty), U)$:*

$$\int_0^t T_{-1}(t-\sigma)Bu(\sigma)d\sigma = \Phi(t)P(s)u.$$

Proposition 5.4 (Composition Property). *For all $t, s \in [0,\infty)$ and $u \in L^p([0,\infty), U)$:*

$$\Phi(t+s)u = T_{-1}(t)\Phi(s)u + \Phi(t)S_l(s)u.$$

Proof. Has been proven last seminar. □

Definition 5.5 (Admissible Control Operator). B is called an *admissible control operator* (for T) if there exists $\tau > 0$ with $\mathrm{ran}(\Phi(\tau)) \subset X$.

Proposition 5.6. *If B is admissible, then for all $t \in [0,\infty)$:*

$$\Phi(t) \in L(L^p([0,\infty), U), X).$$

Proof. $(\beta I - A_{-1})^{-1} \in L(X_{-1}, X)$. Let $u \in L^p([0,\infty), U)$. Then

$$\begin{aligned}
\Phi(\tau)u &= (\beta I - A)(\beta I - A)^{-1}\Phi(\tau)u \\
&= (\beta I - A)(\beta I - A_{-1})^{-1}\Phi(\tau)u \\
&= (\beta I - A)\int_0^\tau (\beta I - A_{-1})^{-1}T_{-1}(\tau - s)Bu(s)ds \\
&= (\beta I - A)\int_0^\tau T_{-1}(\tau - s)\underbrace{(\beta I - A_{-1})^{-1}B}_{\in L(U,X)}u(s)ds \\
&= (\beta I - A)\int_0^\tau T(\tau - s)(\beta I - A_{-1})^{-1}Bu(s)ds.
\end{aligned}$$

Where the final integration is carried out in X, which is possible since $\|\cdot\|$ is stronger than $\|\cdot\|_{-1}$. Therefore $\Phi(\tau)$ is the composition of a closed and a bounded operator and hence closed itself (as an operator with values in X). The closed graph theorem implies, that $\Phi(\tau)$ is bounded. Let $\sigma \in [0,\infty)$ and assume that $\Phi(\sigma) \in L(L^p([0,\infty), U), X)$. Then so is $\Phi(2\sigma)$, because (using the composition property)

$$\Phi(2\sigma) = T_{-1}(\sigma)\Phi(\sigma) + \Phi(\sigma)S_l(\sigma) = T(\sigma)\Phi(\sigma) + \Phi(\sigma)S_l(\sigma).$$

From the above it follows by induction, that $\Phi(2^k\tau)$ is continuous for all $k \in \mathbb{N}$.
Let $\sigma \in [0,\infty)$ and assume that $\Phi(\sigma) \in L(L^p([0,\infty),U),X)$. If $t \in [0,\sigma]$ and $u \in L^p([0,\infty)U)$, then

$$\begin{aligned}\Phi(t)u &= \int_0^t T_{-1}(t-s)Bu(s)ds \\ &= \int_{\sigma-t}^{\sigma} T_{-1}(\sigma-s)Bu(t-\sigma+s)ds \\ &= \Phi(\sigma)S_r(\sigma-t)u.\end{aligned}$$

Which implies that $\Phi(t) \in L(L^p([0,\infty),U),X)$. □

Proposition 5.7. *Let $t, s \in [0,\infty)$ with $t \geq s$. Then $\|\Phi(s)\| \leq \|\Phi(t)\|$.*

Proof. Let $u \in L^p([0,\infty),U)$. Then

$$\begin{aligned}\Phi(t)S_r(t-s)u &= \Phi(s+(t-s))S_r(t-s)u \\ &= T(s)\Phi(t-s)S_r(t-s)u + \Phi(s)S_l(t-s)S_r(t-s)u \\ &= T(s)\Phi(t-s)\underbrace{P(t-s)S_r(t-s)}_{0}u + \Phi(s)\underbrace{S_l(t-s)S_r(t-s)}_{=I}u \\ &= \Phi(s)u.\end{aligned}$$

and so (using $\|S_r(t-s)\| \leq 1$)

$$\|\Phi(s)u\| \leq \|\Phi(t)\|\|u\|.$$

Which in turn implies that $\|\Phi(s)\| \leq \|\Phi(t)\|$. □

Proposition 5.8. *Assume that B is admissible. Then Φ is strongly continuous as a function taking values in $L(L^p([0,\infty),U),X)$.*

Proof. Let $u \in L^p([0,\infty),U)$. For all $t \in [0,1]$:

$$\begin{aligned}\|\Phi(t)u\| &= \|\Phi(t)P(t)u\| \\ &\leq \|\Phi(1)\|\underbrace{\|P(t)u\|}_{\to 0,\ t\to 0}.\end{aligned}$$

Let $t, s \in [0,\infty)$. Then

$$\|\Phi(t+s)u - \Phi(t)u\| = \|T(s)\Phi(t)u + \Phi(s)S_l(t)u - \Phi(t)u\| \leq \underbrace{\|T(s)(\Phi(t)u - \Phi(t)u)\|}_{\to 0, s\to 0} + \underbrace{\|\Phi(s)S_l(t)u\|}_{\to 0, s\to 0}.$$

This implies the strong continuity from above of Φ at t. Let $t, s \in [0,\infty)$ with $s \leq t$. Then

$$\Phi(t) = \Phi(t-s+s) = T(t-s)\Phi(s)u + \Phi(t-s)S_l(s)$$

and so

$$\begin{aligned}\|\Phi(t)u - \Phi(t-s)u\| &= \|T(t-s)\Phi(s)u + \Phi(t-s)(S_l(s)u - u)\| \\ &\leq \sup_{\sigma\in[0,t]}\|T(\sigma)\|\underbrace{\|\Phi(s)u\|}_{\to 0, s\to 0} + \|\Phi(t)\|\underbrace{\|S_l(s)u - u\|}_{\to 0, s\to 0}.\end{aligned}$$

Which proves the strong continuity of Φ from below at t (using strong continuity of S_l). □

Proposition 5.9 (Existence of X Valued Solutions). *Assume that B is admissible. Then for every $x_0 \in X$ and $u \in L^p_{\mathrm{loc}}([0,\infty),U)$ there exists a unique strong solution in X_{-1} to the iIVP associated to (A_{-1},B) with initial value x_0 and input u. Furthermore this solution is in $C([0,\infty),X)$.*

Proof. Let x be the mild solution (in X_{-1}). From last time and the causality property we know for all $s \in [0,\infty)$ and $\forall t \in [0,s]$:

$$x(t) = \underbrace{T_{-1}(t)x_0}_{=T(t)x_0} + \Phi(t)P(s)u$$

and so $x \in C([0,\infty), X)$. In particular this shows that $x \in L^1_{\text{loc}}([0,\infty), Y)$, where $Y := (\text{dom}(A_{-1}), \|\cdot\|_{\text{gr}})$. Since x is the mild solution: $x \in C([0,\infty), X_{-1})$ and for all $t \in [0,\infty)$: $\int_0^t x(s)ds \in \text{dom}(A_{-1})$ and

$$x(t) - x_0 = A_{-1} \int_0^t x(s)ds + \int_0^t Bu(s)ds.$$

Which implies that for all $t \in [0,\infty)$:

$$\begin{aligned} x(t) - x_0 &= \int_0^t A_{-1}x(s)ds + \int_0^t Bu(s)ds \\ &= \int_0^t A_{-1}x(s) + Bu(s)ds, \end{aligned}$$

because $A_{-1} \in L(X, X_{-1}), x \in C([0,\infty), X)$ and $\|\cdot\|$ is stronger than $\|\cdot\|_{-1}$. □

Definition 5.10 (Step Function). Let $\tau > 0$. A function $u \in L^p([0,\infty), U)$ is called a *step function* on $[0,\tau]$ if there exists a partition $0 = t_0 < \cdots < t_n = \tau$ of $[0,\tau]$ and $u_1, \ldots, u_n \in U$ with

$$u = \sum_{i=1}^n \chi_{[t_{i-1},t_i]} u_i.$$

Lemma 5.11 (Step Function Lemma). *Let $\tau > 0$ and $u := \sum_{i=1}^n \chi_{[t_{i-1},t_i]} u_i \in L^p([0,\infty), U)$ a step function on $[0,\tau]$. Then $\Phi(\tau)u \in X$.*

Proof.

$$\begin{aligned} \Phi(\tau)u &= \int_0^\tau T_{-1}(\tau - s)Bu(s)ds \\ &= \sum_{i=1}^n \int_{t_{i-1}}^{t_i} T_{-1}(\tau - s)Bu_i ds \\ &= \sum_{i=1}^n \int_0^{t_i - t_{i-1}} T_{-1}(\tau - t_{i-1} - s)Bu_i ds \\ &= \sum_{i=1}^n T_{-1}(\tau - t_i) \int_0^{t_i - t_{i-1}} T_{-1}(t_i - t_{i-1} - s)Bu_i ds \\ &= \sum_{i=1}^n T_{-1}(\tau - t_i) \underbrace{\int_0^{t_i - t_{i-1}} T_{-1}(s)Bu_i ds}_{\in \text{dom}(A_{-1}) = X} \end{aligned}$$

Using the substitution $\varphi(s) := b - s$ with $b := t_i - t_{i-1}$. □

Proposition 5.12 (Step Function Admissability Criterion). *Let $\tau \in (0,\infty)$ and $M \geq 0$ such that for every step function u on $[0,\tau]$:*

$$\|\Phi(\tau)u\|_X \leq M\|u\|_{L^p}.$$

Then B is admissible.

Proof. Follows at once from the density of step functions in $L^p([0,\tau], U)$, the causality and the fact that $\|\cdot\|_X$ is stronger than $\|\cdot\|_{X_{-1}}$. □

Example 5.13 (Unilateral Right Shift Semigroup with Boundary Control). Let $X := L^2([0,\infty), \mathbb{C})$, $p = 2$, $U := \mathbb{C}$ and T the unilateral right shift semigroup. The adjoint semigroup T^* is the unilateral left shift semigroup. Let $J_X : X \to X'$ be the Riesz isomorphism. Let $i_d : X_1^d \to X$ be the natural injection. Then $(i_d)' \circ J_X$ extends to an anti-linear surjective isometry $J : X_{-1} \to (X_1^d)'$. We have $X_1^d = H^1([0,\infty), \mathbb{C})$ (equality of sets, equivalence of norms). Let $\delta_0 \in (H^1([0,\infty)))'$ be the point evaluation at 0. Define the control operator $B \in L(U, X_{-1})$ by $Bu_0 := u_0 \cdot J^{-1}\delta_0$. Then B is admissible and for all $u \in L^2([0,\infty), U)$ and $t, s \in [0,\infty)$:

$$(\Phi(t)u)(s) = \begin{cases} u(t-s) & s \in [0,t], \\ 0 & \text{else.} \end{cases}$$

Proof. Let $t \in [0, \infty), u \in L^2$ and $f \in H^1([0, \infty))$. Then

$$\begin{aligned}
(J\Phi(t)u)f &= \int_0^t JT_{-1}(t-s)Bu(s)dsf \\
&= \int_0^t JT_{-1}(t-s)u(s)J^{-1}\delta_0 dsf \\
&= \int_0^t \bar{u}(s)(T^*(t-s))'\delta_0 dsf \\
&= \int_0^t \bar{u}(s)\delta_0 T^*(t-s)fds \\
&= \int_0^t \bar{u}(s)f(t-s)ds \\
&= \int_0^t \bar{u}(t-s)f(s)ds \\
&= ((i_d)'J_X(\tilde{u}))f,
\end{aligned}$$

where $\tilde{u} \in X$ is defined by

$$\tilde{u}(s) := \begin{cases} u(t-s) & s \in [0, t], \\ 0 & \text{else.} \end{cases}$$

Therefore $\Phi(t)u = \tilde{u}$, which was to be proven. The substitution with $\varphi : [0, t] \to [0, t]$, $\varphi(s) := t - s$ was used. Then $\varphi' = -1$ and $\varphi(0) = t, \varphi(t) = 0$. ☐

6 Admissible Observation Operators

This section is based on section 4.3 of [TW09]. In this section let:

1. Y a Banach space called the *output space*,
2. $p \in [1, \infty)$,
3. $C \in L(X_1, Y)$ called the *observation operator*,
4. P the truncation operator on $L^p([0, \infty), Y)$.

Definition 6.1 (Reflection Operator). Define the *reflection operator* $R : [0, \infty) \to L(L^p([0, \infty), Y))$ by

$$(R(\tau)f)(t) := \begin{cases} f(\tau - t) & t \in [0, \tau], \\ 0 & \text{else.} \end{cases}$$

Definition 6.2 (Output Map). Define the *extended output map* $\psi_1 \in L(X_1, L^p_{\text{loc}}([0, \infty), Y))$ by

$$(\psi_1 x_0)(t) := CT_1(t)x_0$$

and the *output map* $\Psi_1 : [0, \infty) \to L(X_1, L^p([0, \infty), Y))$ by

$$\Psi_1(\tau)x_0 := P(\tau)\psi_1 x_0.$$

Proposition 6.3 (Reflection Property). *For all $x_0 \in X_1$ and $\tau, \sigma \in [0, \infty)$ with $\sigma \leq \tau$:*

$$\|R(\tau)\Psi_1(\sigma)x_0\| = \|\Psi_1(\sigma)x_0\|.$$

Proposition 6.4 (Dual Composition Property). *Let $\tau, \sigma \in [0, \infty)$ and $x_0 \in X_1$. Then*

$$\psi_1 x_0 = \Psi_1(\tau)x_0 + S_r(\tau)\psi_1 T_1(\tau)x_0$$

and

$$\Psi_1(\tau + \sigma)x_0 = \Psi_1(\tau)x_0 + S_r(\tau)\Psi_1(\sigma)T_1(\tau)x_0.$$

Proof. Let $t \in [0, \infty)$. Then

$$(S_r(\tau)\psi_1 T_1(\tau)x_0)(t) = (S_r(\tau)S_l(\tau)\psi_1 x_0)(t) = \begin{cases} 0 & t \leq \tau, \\ (\psi_1 x_0)(t) & \text{else.} \end{cases}$$

$$\begin{aligned} \Psi_1(\tau+\sigma)x_0 &= P(\tau+\sigma)\psi_1 x_0 \\ &= \Psi_1(\tau)x_0 + P(\tau+\sigma)S_r(\tau)\psi_1 T_1(\tau)x_0 \\ &= \Psi_1(\tau)x_0 + S_r(\tau)\underbrace{P(\sigma)\psi_1}_{\Psi_1(\sigma)} T_1(\tau)x_0. \end{aligned}$$

□

Definition 6.5 (Admissible Observation Operator). C is called an *admissible observation operator* (for T) if there exists $\tau \in [0, \infty)$ such that $\Psi_1(\tau)$ has a (necessarily unique) extension to an operator in $L(X, L^p([0, \infty), Y))$.

Proposition 6.6. *If C is admissible, then for all $t \in [0, \infty)$: $\Psi_1(t)$ has a (necessarily unique) extension to an operator in $L(X, L^p([0, \infty), Y))$.*

Proof. Let $t \in [0, \infty)$ and assume that $\Psi_1(t)$ has an extension. Let $s \in [0, t]$ then $\Psi_1(s) = P(s)\Psi_1(t)$ and so $\Psi_1(s)$ also has an extension. From the dual composition property:

$$\Psi_1(2t) = \Psi_1(t) + S_r(t)\Psi_1(t)T_1(t)$$

and so $\Psi_1(2t)$ also has an extension. □

Definition 6.7. If C is admissible define $\Psi : [0, \infty) \to L(X, L^p([0, \infty), Y))$ by $\Psi(t) :=$ the unique continuous extension of $\Psi_1(t)$.[4]

Example 6.8 (Unilateral Left Shift Semigroup with Boundary Observation). Let $X := L^2([0, \infty), \mathbb{C})$, T the unilateral left shift semigroup on X and $Y := \mathbb{C}$. Then $X_1 = H^1([0, \infty), \mathbb{C})$ (equality of sets, equivalence of norms). Let $C := \delta_0$ be the point evaluation at 0. Then $C \in L(X_1, Y)$ but $C \notin L(X, Y)$. However for all $t \in [0, \infty)$:

$$(\psi_1 f)(t) = \delta_0 T_1(t) f = f(t).$$

Which implies, that for all $\tau \in [0, \infty)$: $\Psi_1(\tau) = P(\tau)|_{X_1}$, where P is the truncation operator on X. Therefore C is admissible and $\Psi = P$.

7 Duality Between Observation and Control

This section is based on section 4.4 of [TW09]. Throughout this section assume that X is a Hilbert space and assume the definitions of section 4.3 have been made (J, X_1^d, etc.).

Definition 7.1. Let Z be a Hilbert space. Let $f \in L(Z, X_{-1})$. Define $f^\sharp \in L(X_1^d, Z)$[5] by letting it be the unique map that satisfies $\forall z \in Z, x_0 \in X_1^d$:

$$J(fz)x_0 = \langle f^\sharp x_0, z\rangle_Z.$$

Such a map exists, because $X_1^d \times Z \ni (x_0, z) \mapsto J(fz)x_0 \in \mathbb{C}$ is sesquilinear and continuous.

In addition let:

1. U a Hilbert space,
2. $B \in L(U, X_{-1})$,
3. Φ the controllability map associated to T and the control operator B (with $p = 2$),
4. Ψ_1 the output map associated to T^* and the observation operator $B^\sharp$ (with $p = 2$).

[4] In [TW09] the output map, its extension and the extended output map are all called Ψ.
[5] In [TW09] $f^\sharp$ is simply denoted f^*

Proposition 7.2 (Duality of Observation and Control). *Let $x_0 \in X_1^d$ and $\tau \in (0,\infty)$. Then for all $t \in [0,\infty)$:*

$$((\Phi(\tau))^\sharp x_0)(t) = \begin{cases} B^\sharp T^*(\tau - t)x_0 & t \in [0,\tau], \\ 0 & \textit{else}. \end{cases} = (R(\tau)\Psi_1(\tau)x_0)(t)$$

In particular

$$\|(\Phi(\tau))^\sharp x_0\| = \|\Psi_1(\tau)x_0\|.$$

If B is admissible for T and we view $\Phi(\tau) \in L(L^2([0,\infty),U),X)$, then $(\Phi(\tau))^*$ extends $(\Phi(\tau))^\sharp$.

Proof. Let $u \in L^2([0,\infty),U)$ and $x_0 \in X_1^d$. Then (because $T_{-1} = J^{-1} \circ (T^*|_{X_1^d})' \circ J$)

$$\begin{aligned} J(\Phi(\tau)u)x_0 &= \int_0^\tau J(T_{-1}(\tau-\sigma)Bu(\sigma))x_0 d\sigma \\ &= \int_0^\tau J(Bu(\sigma))T^*(\tau-\sigma)x_0 d\sigma \\ &= \int_0^\tau \langle\langle B^\sharp T^*(\tau-\sigma)x_0, u(\sigma)\rangle_U d\sigma \\ &= \langle v, u\rangle_{L^2}, \end{aligned}$$

where $v \in L^2$ is defined by

$$v(t) := \begin{cases} B^\sharp T^*(\tau - t)x_0 & t \in [0,\tau], \\ 0 & \text{else}. \end{cases}$$

If B is admissible, then $\Phi(\tau)u \in X$ and so

$$J(\Phi(\tau)u)x_0 = \langle x_0, \Phi(\tau)u\rangle_X = \langle(\Phi(\tau))^* x_0, u\rangle_{L^2}.$$

□

Theorem 7.3 (Duality of Admissability Concepts). *B is an admissible control operator for T if and only if $B^\sharp$ is an admissible observation operator for T^*.*

Proof. Assume that B is admissible. Let $\tau \in [0,\infty)$. Then $\Phi(\tau) \in L(L^2([0,\infty),U),X)$ and so $(\Phi(\tau))^* \in L(X, L^2([0,\infty),U))$. Now for all $x_0 \in X_1^d$:

$$\|\Psi_1(\tau)x_0\|_{L^2} = \|(\Phi(\tau))^* x_0\|_{L^2} \leq \|(\Phi(\tau))^*\|\|x_0\|_X.$$

Which implies that $\Psi_1(\tau)$ can be extended.
Assume that $B^\sharp$ is an admissible observation operator. Let $\tau \in (0,\infty)$. Then for all $x_0 \in X_1^d$:

$$\|\Psi_1(\tau)x_0\|_{L^2} \leq \|\Psi(\tau)\|\|x_0\|_X.$$

Let $u \in L^2([0,\infty),U)$ be a step function on $[0,\tau]$. Then for all $x_0 \in X_1^d$:

$$\langle x_0, \Phi(\tau)u\rangle_X = J(\Phi(\tau)u)x_0 = \langle(\Phi(\tau))^\sharp x_0, u\rangle_{L^2} = \langle R(\tau)\Psi_1(\tau)x_0, u\rangle_{L^2}$$

and so

$$|\langle\Phi(\tau)u, x_0\rangle_X| \leq \|R(\tau)\Psi_1(\tau)x_0\|\|u\|_{L^2} \leq \|\Psi(\tau)\|\|x_0\|_X\|u\|_{L^2}$$

which implies by density of X_1^d in X that

$$\|\Phi(\tau)u\|_X \leq \|\Psi(\tau)\|\|u\|_{L^2}.$$

The step function admissability criterion concludes that B is admissible. □

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