Linear-Quadratic Optimal Control for Discrete Time Systems

Jannik Daun

PHSymposium, 2024

Contents

Contents

Contents

Definition (Discrete-time system)

- **Discrete-time system** $\Sigma := (\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}) \in L(\mathfrak{X}) \times L(\mathcal{U}, \mathfrak{X}) \times L(\mathfrak{X}, \mathcal{Y}) \times L(\mathcal{U}, \mathcal{Y})$, where $\mathfrak{X}, \mathcal{U}, \mathcal{Y}$ are Hilbert spaces.
- **State map** $z_d : \mathbb{N} \times \mathfrak{X} \times \mathcal{U}^{\mathbb{N}} \longrightarrow \mathfrak{X}$,
$$z_d(0, z_0, u_d) := z_0,$$
$$z_d(j+1, z_0, u_d) := \mathcal{A} z_d(j, z_0, u_d) + \mathcal{B} u_d(j).$$
- **Output map** $y_d : \mathbb{N} \times \mathfrak{X} \times \mathcal{U}^{\mathbb{N}} \longrightarrow \mathcal{Y}$,
$$y_d(j, z_0, u_d) := \mathcal{C} z_d(j, z_0, u_d) + \mathcal{D} u_d(j).$$

Proposition (Discrete-time Duhamel's principle)

$$\forall z_0 \in \mathfrak{X}, u_d \in \mathcal{U}^{\mathbb{N}}, j \in \mathbb{N} : z_d(j, z_0, u_d) = \mathcal{A}^j z_0 + \sum_{k=0}^{j-1} \mathcal{A}^k \mathcal{B} u_d(j-1-k).$$

Contents

Definition (Discrete-time cost functional)

$$J_d : \mathfrak{X} \times \mathcal{U}^{\mathbb{N}} \longrightarrow [0, \infty],$$

$$J_d(z_0, u_d) := \sum_{j=0}^{\infty} \|y_d(j, z_0, u_d)\|_{\mathcal{Y}}^2 + \|u_d(j)\|_{\mathcal{U}}^2.$$

Discrete-time Linear-Quadratic (LQ) optimal control problem

For all $z_0 \in \mathfrak{X}$ prove existence and uniqueness of a minimizer of $J_d(z_0, \bullet)$ and find an explicit formula for it.

Definition (Optimizability)

Σ is called **optimizable** if

$$\forall z_0 \in \mathfrak{X} \, \exists u_d \in \mathcal{U}^{\mathbb{N}} : J_d(z_0, u_d) < \infty.$$

Theorem (Existence and uniqueness of the optimal control)

Assume that Σ is optimizable. Then for every $z_0 \in \mathfrak{X}$ there exists a unique element of $\ell^2(\mathbb{N},\mathcal{U})$ denoted $u_d^{\mathrm{opt}}(\bullet, z_0)$ with

$$J_d(z_0, u_d^{\mathrm{opt}}(\bullet, z_0)) = \inf_{v \in \mathcal{U}^{\mathbb{N}}} J_d(z_0, v).$$

Definition (Optimal output/state)

For $z_0 \in \mathfrak{X}$ let $y_d^{\mathrm{opt}}(\bullet, z_0)$ the output and $z_d^{\mathrm{opt}}(\bullet, z_0)$ the state with input $u_d^{\mathrm{opt}}(\bullet, z_0)$ and initial value z_0.

Proposition (Continuity and linearity of the optimal control)

Assume that Σ is optimizable. Then:

- *The map*
$$\mathcal{I} : \mathfrak{X} \longrightarrow \ell^2(\mathbb{N},\mathcal{U}) \times \ell^2(\mathbb{N},\mathcal{Y}),$$
$$\mathcal{I}(z_0) := (u_d^{\mathrm{opt}}(\bullet, z_0), y_d^{\mathrm{opt}}(\bullet, z_0))$$
is linear and bounded.
- *The map $\Pi := \mathcal{I}^*\mathcal{I} \in L(\mathfrak{X})$ is self-adjoint, nonnegative and $\forall z_0 \in \mathfrak{X}$:*
$$\langle \Pi z_0, z_0 \rangle_{\mathfrak{X}} = \langle \mathcal{I} z_0, \mathcal{I} z_0 \rangle = J_d(z_0, u^{\mathrm{opt}}(\bullet, z_0)).$$

Definition (Feedback operator)

For $P \in L(\mathfrak{X})$ let

$$Q(P) := \mathcal{D}^*\mathcal{D} + I + \mathcal{B}^* P \mathcal{B}, \quad R(P) := \mathcal{D}^*\mathcal{C} + \mathcal{B}^* P \mathcal{A}, \quad \mathcal{F}(P) := -Q(P)^{-1} R(P).$$

The operator $\mathcal{F}(P)$ is called the **feedback operator** associated to P.

Theorem (Optimal control by state feedback)

Assume that Σ is optimizable.

Then $\forall z_0 \in \mathfrak{X}, j \in \mathbb{N}$:

$$z_d^{\mathrm{opt}}(j, z_0) = (\mathcal{A} + \mathcal{B}\mathcal{F}(\Pi))^j z_0,$$
$$u^{\mathrm{opt}}(j, z_0) = \mathcal{F}(\Pi) z_d^{\mathrm{opt}}(j, z_0).$$

Lemma (Bellman's principle of optimality)

Assume Σ optimizable. Let $z_0 \in \mathfrak{X}$, $u_0 \in \mathcal{U}$, $z_1 := \mathcal{A}z_0 + \mathcal{B}u_0$ and $y_0 := \mathcal{C}z_0 + \mathcal{D}u_0$. Then

$$\langle z_0, \Pi z_0 \rangle_{\mathfrak{X}} \leq \|y_0\|^2_{\mathcal{Y}} + \|u_0\|^2_{\mathcal{U}} + \langle z_1, \Pi z_1 \rangle_{\mathfrak{X}}$$

with equality if and only if $u_0 = u^{\mathrm{opt}}(0, z_0)$ and in that case for all $j \in \mathbb{N}$:

$$u_d^{\mathrm{opt}}(j+1, z_0) = u_d^{\mathrm{opt}}(j, z_1).$$

Lemma (Key lemma)

Let $P \in L(\mathfrak{X})$ self-adjoint and nonnegative. Define

$$G : \mathfrak{X} \times \mathcal{U} \longrightarrow [0, \infty),$$

$$G(z_0, u_0) := \|u_0\|^2_{\mathcal{U}} + \|y_0\|^2_{\mathcal{Y}} + \langle z_1, P z_1 \rangle_{\mathfrak{X}},$$

where $z_1 := \mathcal{A}z_0 + \mathcal{B}u_0$ and $y_0 := \mathcal{C}z_0 + \mathcal{D}u_0$.
Then for every $z_0 \in \mathfrak{X}$: $G(z_0, \bullet)$ has a unique global minimum at $\mathcal{F}(P)z_0$ and

$$G(z_0, \mathcal{F}(P)z_0) = \langle z_0, \mathcal{C}^*\mathcal{C} + \mathcal{A}^* P \mathcal{A} - R(P)^* Q(P)^{-1} R(P) z_0 \rangle_{\mathfrak{X}}.$$

Contents

Definition (CARE)

$P \in L(\mathfrak{X})$ is called a solution to the **control algebraic Riccati equation** (CARE) associated to Σ if

$$P = \mathcal{C}^*\mathcal{C} + \mathcal{A}^* P \mathcal{A} - R(P)^* Q(P)^{-1} R(P).$$

Proposition (Characterisation of optimizability via the CARE)

- *If Σ is optimizable, then Π is a solution to the CARE.*
- *If the CARE has a self-adjoint and nonnegative solution, then Σ is optimizable.*

([OC04], Lemma 3.3)

Definition (FARE)

$P \in L(\mathfrak{X})$ is called a solution to the **filter algebraic Riccati equation** (FARE) associated to Σ if P is a solution to the CARE associated to the **adjoint system** $(\mathcal{A}^*, \mathcal{C}^*, \mathcal{B}^*, \mathcal{D}^*)$.

Theorem (Sufficient condition for uniqueness of solution to the CARE)

Assume that the FARE has a self-adjoint and nonnegative solution.
Let $\Pi_0 \in L(\mathfrak{X})$ be a self-adjoint and nonnegative solution to the CARE with

$$r(\mathcal{A} + \mathcal{B}\mathcal{F}(\Pi_0)) < 1,$$

where r denotes the spectral radius.
Then Π_0 is the unique self-adjoint and nonnegative solution to the CARE.

([OC04], Lemma 3.7)

Proposition (Reduction to finite dimension)

Let:

- $\mathfrak{X} := L^2([0,1], \mathbb{C}^n)$, $\mathcal{U} := L^2([0,1], \mathbb{C}^p)$, $\mathcal{Y} := L^2([0,1], \mathbb{C}^m)$,
- $A_d \in \mathbb{C}^{n\times n}, B_d \in \mathbb{C}^{n\times p}, C_d \in \mathbb{C}^{m\times n}, D_d \in \mathbb{C}^{m\times p}$, $P_d \in \mathbb{C}^{n\times n}$,
- $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}, \mathcal{P} :=$ *acting by multiplication with* A_d, B_d, C_d, D_d, P_d *on* L^2.

Then:

- *If* P_d *is a self-adjoint, nonnegative solution to the CARE/FARE associated to* (A_d, B_d, C_d, D_d)*, then* $\mathcal{P}$ *is a self-adjoint, nonnegative solution to the CARE/FARE associated to* $(\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D})$*,*
- *The spectral radius of* $A_d + B_d\mathcal{F}(P_d)$ *(as a matrix) is equal to the spectral radius of* $\mathcal{A} + \mathcal{B}\mathcal{F}(\mathcal{P})$*.*

References

For more on discrete-time systems see [OC04] and [OS08].

[OC04] Mark R. Opmeer and Ruth F. Curtain. "Linear Quadratic Gaussian Balancing for Discrete-Time Infinite-Dimensional Linear Systems". In: *SIAM Journal on Control and Optimization* 43.4 (2004), pp. 1196–1221.

[OS08] Mark R. Opmeer and Olof J. Staffans. "Optimal state feedback input-output stabilization of infinite-dimensional discrete time-invariant linear systems". In: *Complex Anal. Oper. Theory* 2.3 (2008), pp. 479–510.