Project 8: Maximal Equicontinuous Factor

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Definition (Topological Dynamical System (TDS))

Let (X, d) be a **compact** metric space and T a topological group and let there be a continuous map $T \times X \to X$ $((t, x) \mapsto tx)$ such that for all $s, t \in T$ and $x \in X$

- $ex = x$
- $(st)x = s(tx)$.

Then we call (X, T) a **topological dynamical system** (TDS).

Remark

Note that for each $t \in T$ the map $x \mapsto tx$ is a homeomorphism.

Remark

Let $\alpha : X \to X$ be a homeomorphism. Then this gives rise to a $\mathbb{Z}$ action with $nx := \alpha^n(x)$ for $n \in \mathbb{Z}$. In this case we write (X, α) for this dynamical system.

Definition (Factors and Isomorphisms)

A TDS (Y, T) is a **factor** of (X, T) if there exists a **factor map**, i.e. a surjective and continuous map

$$\pi : X \to Y$$

such that $\pi(tx) = t\pi(x)$ for all $t \in T,\ x \in X$.
A factor map is a **conjugacy** (of TDS) if it is a homeomorphism.
Two TDS are conjugated if there exists an conjugacy between them.

Definition (ICER and Quotient)

An **invariant closed equivalence relation** (ICER) $R \subseteq X \times X$ is a equivalence relation, such that R is closed and $tR = R$ for all $t \in T$.
Then the quotient X/R becomes a TDS with $t[x] := [tx]$ for all $t \in T$ and $x \in X$.

Proposition

Let $R \subseteq X \times X$ be an ICER. Then we can define a factor by

$$\pi_R : X \to X/R \quad \pi_R(x) := [x] \quad \forall x \in X.$$

Let $\pi : X \to Y$ be a factor map. Then we can define an ICER by

$$R_\pi := \{(x, y) \in X \times X \mid \pi(x) = \pi(y)\}.$$

In this case we have $(Y, T) \cong (X/R_\pi, T)$.

Definition (Conjugated and Equivalent factors)

Let (Y, T) and (Z, T) be two factors of (X, T) with factor maps π and ρ.
The factors are **equivalent** if $R_\pi = R_\rho$.
The factors are **conjugated** if $(Y, T) \cong (Z, T)$.

Definition (Equicontinuous TDS)

The TDS (X, T) is **equicontinuous** if for $\varepsilon > 0$ there is some $\delta > 0$ such that

$$d(x, y) < \delta \Rightarrow d(tx, ty) < \varepsilon$$

for all $x, y \in X$ and $t \in T$.

Remark

Intuitively equicontinuity means that two points that are close, were always and will always be close.

Definition (Equicontinuous TDS)

The TDS (X, T) is **equicontinuous** if for $\varepsilon > 0$ there is some $\delta > 0$ such that

$$d(x, y) < \delta \Rightarrow d(tx, ty) < \varepsilon$$

for all $x, y \in X$ and $t \in T$.

Proposition

Let $d, \tilde{d}$ be two metrics on X, that induce the same topology. Then (X, T) is equicontinuous with respect to d if and only if it is equicontinuous with respect to $\tilde{d}$.

Proposition

For an equicontinuous action we can always choose a metric, inducing the same topology, that is invariant under the action.

Definition (Equicontinuous TDS)

The TDS (X, T) is **equicontinuous** if for $\varepsilon > 0$ there is some $\delta > 0$ such that

$$d(x, y) < \delta \Rightarrow d(tx, ty) < \varepsilon$$

for all $x, y \in X$ and $t \in T$.

Proposition

Equicontinuity is preserved under conjugacies, (countable) products and subsystems.

Example: Circle Rotations $\mathbb{Z}$-action

We look at the circle $\mathbb{T} := \mathbb{R}/\mathbb{Z}$ (identify $[x] = x + \mathbb{Z}$ for $x \in \mathbb{R}$ with $e^{2\pi i x}$ in $\mathbb{C}$). The metric on $\mathbb{T}$ is given by

$$d_{\mathbb{T}}([x],[y]) := \min_{n\in\mathbb{Z}} |x - y + n|.$$

Fix $a \in \mathbb{R}$ and define

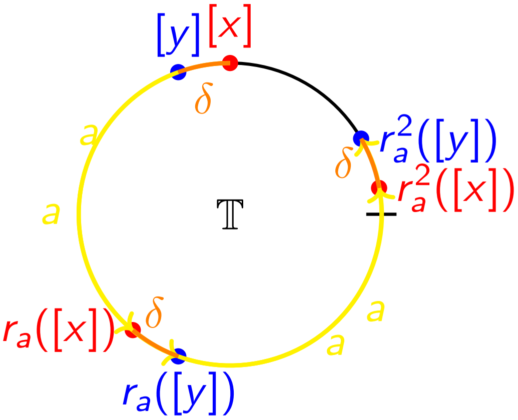
$$r_a([x]) := [x + a].$$

The TDS $(\mathbb{T}, r_a)$ is minimal iff $a \notin \mathbb{Q}$.

We have for $[x],[y] \in \mathbb{S}$ and $n \in \mathbb{Z}$ that

$$d_{\mathbb{T}}(r_a^n([x]), r_a^n([y])) = d_{\mathbb{T}}([x],[y]),$$

i.e. r_a is isometric. Equicontinuity is an immediate consequence.



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Let (X, T) be a TDS.

Definition (maximal equicontinuous factor (MEF))

A factor $\pi : (X, T) \to (X_{\mathrm{MEF}}, T)$ is a **MEF** of (X, T) if and only if

- (X_{MEF}, T) is equicontinuous,
- π is maximal: $\forall \varphi : (X, T) \to (Y, T)$ factor s.t. (Y, T) equicontinuous: $R_\pi \subseteq R_\varphi$.

Theorem (existence and uniqueness of the MEF)

(X, T) has a MEF that is unique up to equivalence.

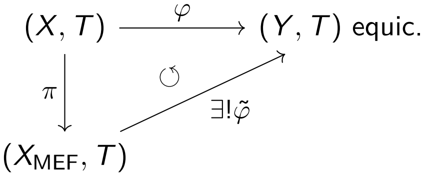
Example (MEF of equicontinuous TDS is original TDS)

(X, T) equicontinuous, then $I : (X, T) \to (X, T)$ is the MEF of (X, T).

Proposition (universal property of the MEF)

Let $\pi : (X, T) \to (X_{MEF}, T)$ *be the MEF.*

$$\forall\, \varphi : (X, T) \to (Y, T) \text{ factor with } (Y, T) \text{ equicontinuous} :$$
$$\exists!\, \text{factor } \tilde{\varphi} : (X_{MEF}, T) \to (Y, T) : \varphi = \tilde{\varphi} \circ \pi$$



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(X, T) TDS with T abelian, $T^\sharp$ (T^*) the set of all (continuous) characters of T.

Definition (Koopman representation)

$$U : T \longrightarrow \mathscr{L}(C(X))$$
$$(U(t)f)(x) := f(tx)$$

Remark

T-action continuous on $T \times X \Rightarrow U$ strongly continuous T-representation

(for proof see [EFHN15], Theorem 4.17)

Definition (eigenvalue, eigenfunction)

$0 \neq f \in C(X)$ *eigenfunction* of U to *eigenvalue* $\chi \in T^*$

$$:\Leftrightarrow f \in \ker(U - \chi) := \bigcap_{t \in T} \ker(U(t) - \chi(t)I)$$

Definition (discrete spectrum)

(X, T) has (TDS) **discrete spectrum** $:\Leftrightarrow C(X) = \overline{\text{lin}} \bigcup_{\chi \in T^*} \ker(U - \chi)$

Theorem (equicontinuity $\Leftrightarrow$ discrete spectrum)

(X, T) is equicontinuous $\Leftrightarrow$ (X, T) has discrete spectrum

(for proof see [HK23], Theorem 2.11)

Theorem (MEF eigenfunction characterisation)

Let $\pi : (X, T) \to (X_{\text{MEF}}, T)$ be the MEF. Then $\forall x_1, x_2 \in X$:

$$\pi(x_1) = \pi(x_2) \Leftrightarrow \forall f \in \bigcup_{\chi \in T^*} \ker(U - \chi) : f(x_1) = f(x_2)$$

Intermezzo: Duality of categories!

- $\mathbf{C}$:= category of commutative unital C^*-algebras,
- $\mathbf{D}$:= category of compact Hausdorff spaces,
- contravariant Gelfand functor $G : \mathbf{C} \to \mathbf{D}$,
$$G(\mathscr{A}) := \{\varphi : \mathscr{A} \to \mathbb{C} \mid \varphi \text{ morphism}\} \text{ with the weak-* topology},$$
$$G(\Psi : \mathscr{A} \to \mathscr{B}) : G(\mathscr{B}) \to G(\mathscr{A}),\ \varphi \mapsto \varphi \circ \Psi,$$
- contravariant continuous function functor $C : \mathbf{D} \to \mathbf{C}$, $X \mapsto C(X)$,
$$C(f : X \to Y) : C(Y) \to C(X),\ g \mapsto g \circ f,$$
- evaluation map $\mathrm{eval} : I \Rightarrow GC$, $\mathrm{eval}(x)f := f(x)$ is natural isomorphism,
- Gelfand transform $\widehat{\ } : I \Rightarrow CG$, $\widehat{a}(\varphi) := \varphi(a)$ is natural isomorphism.

MEF via C^*-algebras

- $\mathscr{A} := \overline{\text{lin}} \bigcup_{\chi \in T^*} \ker(U - \chi) \subseteq C(X)$,
- $\mathscr{A}$ is smallest unital C^*-subalgebra containing all eigenfunctions of U,
- $\mathscr{A}$ U-invariant $\to$ T-action on $G(\mathscr{A})$:
$$t \cdot \varphi := G(U(t))\varphi,$$
- $\iota : \mathscr{A} \to C(X)$ natural injection,
- $G(\iota) : G(C(X)) \to G(\mathscr{A})$,
- MEF of (X, T) is
$$\pi : (X, T) \to (G(\mathscr{A}), T),$$
$$\pi := G(\iota) \circ \text{eval},$$

- $\widehat{\ } : I \Rightarrow CG$ natural:
$$\begin{array}{ccc} \mathscr{A} & \xrightarrow{\ \widehat{\ }\ } & C(G(\mathscr{A})) \\ {\scriptstyle\iota}\downarrow & \circlearrowleft & \downarrow{\scriptstyle C(G(\iota))} \\ C(X) & \xrightarrow{\ \widehat{\ }\ } & C(G(C(X))) \end{array}$$
- $G(\iota)$ surjective $\Leftrightarrow$ $C(G(\iota))$ injective $\Leftrightarrow$ ι injective,
- let $x_1, x_2 \in X$:
$\pi(x_1) = \pi(x_2) \Leftrightarrow \forall f \in \mathscr{A} : f(x_1) = f(x_2)$
$$\Leftrightarrow \forall f \in \bigcup_{\chi \in T^*} \ker(U - \chi) : f(x_1) = f(x_2).$$

Contents

MEF vs. Kronecker subsystem

T still abelian

(X, T) TDS

- Koopman rep. $U : T \to \mathscr{L}(C(X))$,
- (X, T) has discrete spectrum $\Leftrightarrow \overline{\text{lin}} \bigcup_{\chi \in T^*} \ker(U - \chi) = C(X)$,
- MEF: associated to the U-invariant C^*-subalgebra
$$\overline{\text{lin}} \bigcup_{\chi \in T^*} \ker(U - \chi) \subseteq C(X).$$

(Y, T) measure preserving system (MPS)

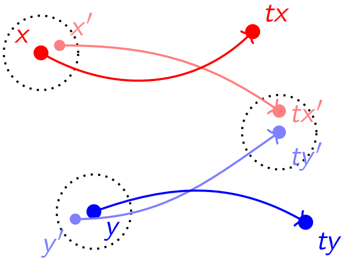
- Koopman rep. $V : T \to \mathscr{L}(L^2(Y))$,
- (Y, T) has discrete spectrum $\Leftrightarrow \overline{\text{lin}} \bigcup_{\chi \in T^\sharp} \ker(V - \chi) = L^2(Y)$,
- Kronecker subsystem: associated to the V-invariant Markov sublattice
$$\overline{\text{lin}} \bigcup_{\chi \in T^\sharp} \ker(V - \chi) \subseteq L^2(Y).$$

MEF is the TDS analogue of the Kronecker subsystem!

Contents

Definition (regionally proximal relation)

Let (X, T) be a TDS. Two points $x, y \in X$ are called *regionally proximal* if for each $\varepsilon > 0$ there exist $x' \in B_\varepsilon(x), y' \in B_\varepsilon(y)$ and $t \in T$ such that $d(tx', ty') < \varepsilon$. We denote the set of regionally proximal pairs by $Q_2(X) \subseteq X \times X$.



Definition (regionally proximal relation)

Let (X, T) be a TDS. Two points $x, y \in X$ are called *regionally proximal* if for each $\varepsilon > 0$ there exist $x' \in B_\varepsilon(x), y' \in B_\varepsilon(y)$ and $t \in T$ such that $d(tx', ty') < \varepsilon$. We denote the set of regionally proximal pairs by $Q_2(X) \subseteq X \times X$.

Equivalently: There is (x_n), (y_n) in X and (t_n) in T, s.t.

$$x_n \to x, \;\; y_n \to y \text{ and } d(t_n x_n, t_n y_n) \to 0.$$

and

$$Q_2(X) = \bigcap_{U \in \mathcal{U}(\Delta(X))} \overline{TU}$$

with $\Delta(X) = \{(x, x) \mid x \in X\}$ the diagonal in $X \times X$.

Theorem

The regionally proximal relation is closed, invariant, symmetric and reflexive.

1. symmetric and reflexive by definition
2. closed, since intersection of closed sets
3. invariant: $(x, y) \in Q_2(X)$, i.e. $d(t_n x_n, t_n y_n) \to 0$ then $d(t_n t_0^{-1} t_0 x_n, t_n t_0^{-1} t_0 y_n) \to 0$.

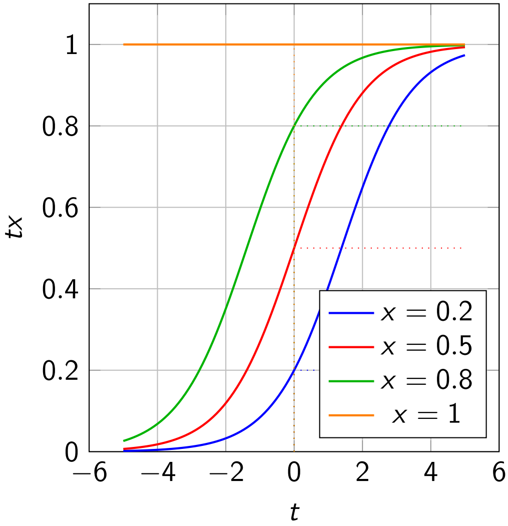
Not transitive in general! See Example.

Let $X = [0,2]$ and $T = \mathbb{R}$. Then the following map is an $\mathbb{R}$-action on $[0,2]$ and defines a TDS.

$$\mathbb{R} \times [0,2] \to [0,2]$$

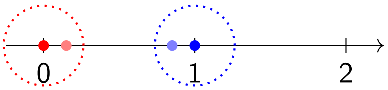
$$(t,x) \mapsto tx := \begin{cases} 0 & x = 0 \\ \frac{1}{1+(\frac{1}{x}-1)e^{-t}} & x \in (0,1] \\ \frac{1}{1+(\frac{1}{x-1}-1)e^{-t}} + 1 & x \in (1,2] \end{cases}$$

Every point greater 0 and smaller 1 tends to 1 and every point greater 1 tends to 2 with $t \to \infty$.
We also observe that $\{0\}$, $\{1\}$ and $\{2\}$ are closed invariant.

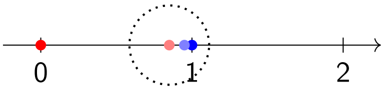


Consider the system from the previous slide.
The pairs $(0, 1)$ and $(1, 2)$ are regionally proximal.
But $(0, 2)$ is not a regionally proximal pair.

For 0 and 1 we have:



With t large enough:



Proposition

Let $\varphi : (X, T) \to (Y, T)$ a factor. Then $(\varphi \times \varphi)(Q_2(X)) \subseteq Q_2(Y)$.

Theorem

Let (X, T) a TDS and $\varphi : (X, T) \to (Y, T)$ an equicontinuous factor. Let R_φ the ICER generated by φ. Then $Q_2(X) \subseteq R_\varphi$.

Proof by contradiction using the invariant metric.

Corollary

Let (X, T) be an equicontinuous TDS. Then $Q_2(X) = \Delta(X)$.

The identity map with $\Delta(X)$ as ICER is an equicontinuous factor.

Since the MEF is an equicontinuous factor, we have

Corollary

Let (X, T) be a TDS and R_{MEF} the ICER generated by the MEF. Then $Q_2(X) \subseteq R_{MEF}$.

Do we also have $R \subseteq Q_2(X)$?

But first:

Definition (topologically weakly mixing)

Let (X, T) be a TDS. Then (X, T) is called *weakly mixing* if every non-empty, T-invariant and open $U \subseteq X \times X$ is dense in $X \times X$.

Proposition (weakly mixing systems have trivial Q_2)

Let (X, T) be a weakly mixing TDS. Then $Q_2(X) = X \times X$.

Using the theorem of Baire and the characterization of the regionally proximal equation using intersections.

Corollary (MEF of weakly mixing system)

The MEF of a weakly mixing system is trivial.

Since $Q_2(X) = X \times X \subseteq R_{\text{MEF}}$.

(X, T) equicontinuous

1. MEF is the identity map, biggest possible factor
2. $Q_2(X) = \Delta(X)$, smallest possible reflexive relation

(X, T) weakly mixing

1. MEF is one-point system, smallest possible factor
2. $Q_2(X) = X \times X$, biggest possible relation

We have $Q_2(X) = R_{\mathrm{MEF}}$ with some assumptions.

Theorem

Let (X, T) be a minimal TDS with an invariant Borel probability measure and R_{MEF} be the ICER associated with the MEF of X. Then $Q_2(X) = R_{MEF}$.

Note that minimality is important for $Q_2(X)$ being an equivalence relation (see Example). For a proof see [Aus88, p.130, Thm. 8.].
The condition of having an invariant Borel probability measure is always satisfied for abelian groups.

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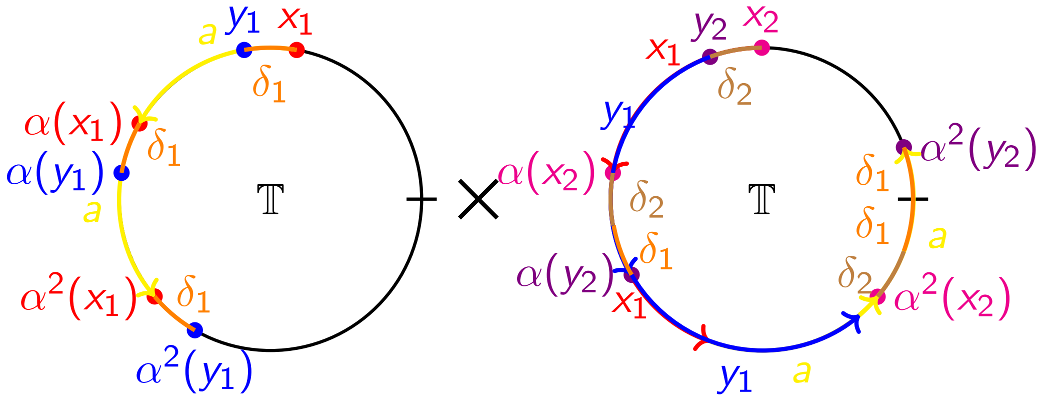
Fix $a \in \mathbb{R} \setminus \mathbb{Q}$ and look at $\mathbb{T}^2 = \mathbb{R}/\mathbb{Z} \times \mathbb{R}/\mathbb{Z}$ with the skew rotation $\alpha : \mathbb{T}^2 \to \mathbb{T}^2$ with

$$\alpha([x_1], [x_2]) := ([x_1 + a], [x_2 + x_1]).$$

The arising TDS $(\mathbb{T}^2, \mathbb{Z})$ is minimal and not equicontinuous.



Example

The MEF of the skew-ration $(\mathbb{T}^2, \alpha)$ is given by the circle rotation $(\mathbb{T}, r_a)$ with factor map π given by $([x_1], [x_2]) \mapsto [x_1]$.

We need to prove that

$$Q_2(\mathbb{T}^2) = R_\pi = \{(([x_1], [x_2]), ([y_1], [y_2])) \in \mathbb{T}^2 \times \mathbb{T}^2 | \; [x_1] = [y_1]\}.$$

We already know that $(\mathbb{T}, r_a)$ is equicontinuous, which implies $Q_2(\mathbb{T}^2) \subseteq R_\pi$.

For the other direction we prove that $(([z],[x]),([z],[y])) \in R_\pi$ is regional proximal.

Fix $\varepsilon > 0$. W.l.o.g. $0 \leq x \leq y < 1$. Choose some $0 < b < \frac{\varepsilon}{2}$ and $n \in \mathbb{N}_0$ such that

$$0 \leq (y-x) - nb < \frac{\varepsilon}{2}.$$

Then $([z+b],[x])$ and $([z],[y])$ are in the ε balls around $([z],[x])$ and $([z],[y])$ and

$$d_{\mathbb{T}^2}(\alpha^n([z+b],[x]),\alpha^n([z],[y]))$$

$$=d_{\mathbb{T}^2}(([z+b+na],[x+n(z+b)+\frac{n(n+1)}{n}a]),([z+na],[y+nz+\frac{n(n+1)}{2}a]))$$

$$\leq b+x-y-nb < \frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon$$

Remark

Note that for $([z],[x]) \in \mathbb{T}^2$ and $n \in \mathbb{N}$ we have

$$\alpha^n([z],[x]) = ([z+na],[x+nz+\frac{n(n-1)}{2}a]).$$

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Definition (m-Equicontinuity [GL25; Jin05])

Fix $m \in \mathbb{N}$. Then (X, T) is *m-equicontinuous* if for all $\varepsilon > 0$ there exists a $\delta > 0$, such that for all $U \subseteq X$ open with $\text{diam}(U) < \delta$, $x_1, \ldots, x_m \in U$ and $t \in T$ there are $i, j \in \{1, \ldots, m\}$ with $i \neq j$ such that $d(tx_i, tx_j) < \varepsilon$.

Proposition

If (X, T) is m-equicontinuous for some $m \in \mathbb{N}$ it is also m'-equicontinuous for all $m' > m$.

Proof is clear by definition.
Note that the "normal" equicontinuity is 2-equicontinuity.

Definition (m-MEF)

Fix $m \in \mathbb{N}$ with $m \geq 2$. Assume that (X, T) is a minimal TDS and let its MEF be given by $\pi_{\mathrm{MEF}} : X \to X_{\mathrm{MEF}}$. Let $R_{\pi_{\mathrm{MEF}}}$ be the ICER generated by π_{MEF}. Let $\pi : (X, T) \to (Y, T)$ be a factor and R_π the ICER generated by π. Then π is called a *m-MEF* if

$$\forall A \in X/R_{\pi_{\mathrm{MEF}}} \exists B_1, \ldots, B_{m-1} \in X/R_\pi : A = \bigcup_{i=1}^{m-1} B_i.$$

Note that the indices of the B_i only go up to $m-1$! As before, we have:

Remark

Any m-MEF is also an m'-MEF for all $m' > m$ since the B_i in the definition of an m-MEF need not be pairwise distinct.

Importantly, we have

Proposition (Lemma 4.14 in [GL25])

Fix $m \in \mathbb{N}$ with $m \geq 2$. Let $\pi : (X, T) \to (Y, T)$ be an m-MEF. Then (Y, T) is m-equicontinuous.

Generalizations

We can generalize

1. the regionally proximal relation to an m-regionally proximal relation $Q_m(X) \subset X^m$.
2. the diagonal to
$\Delta^m(X) := \{(x_1, \ldots, x_m) \in X^m \mid \text{there are } i, j \in \{1, \ldots, m\} \text{ such that } x_i = x_j\}$.

And get similarly to before

Theorem (Theorem 4.6 in [GL25])

Let (X, T) be a minimal TDS. Then (X, T) is m-equicontinuous if and only if $Q_m(X) \setminus \Delta^m(X) = \varnothing$ for $2 \leq m \in \mathbb{N}$.

Theorem ([GL25])

Let (X, T) be a minimal TDS and $\pi : X \to X_{MEF}$ its MEF. Then (X, T) is m-equicontinuous if and only if $\#\pi^{-1}(\{y\}) < m$ for all $y \in X_{MEF}$.

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Let σ be the left shift on $\{0,1\}^{\mathbb{Z}}$. Then $(\{0,1\}^{\mathbb{Z}}, \sigma)$ is a TDS.

Example (Thue Morse Subshift)

Take the substitution
$$\begin{cases} 0 \mapsto 01 \\ 1 \mapsto 10 \end{cases}.$$
It has a fixed point $v \in \{0,1\}^{\mathbb{Z}}$ and gives a minimal subshift
$$X := \overline{\{\sigma^n v \mid n \in \mathbb{Z}\}}.$$

Example (Period Doubling Subshift)

Take the substitution
$$\begin{cases} 0 \mapsto 01 \\ 1 \mapsto 00 \end{cases}.$$
It has a fixed point $w \in \{0,1\}^{\mathbb{Z}}$ and gives a minimal subshift
$$Y := \overline{\{\sigma^n w \mid n \in \mathbb{Z}\}}.$$

Then there is a 2-to-1 factor $\pi : X \to Y$ defined by
$$\pi((a_n)_{n\in\mathbb{Z}}) := \begin{cases} 0 & ; a_n \neq a_{n+1} \\ 1 & ; a_n = a_{n+1} \end{cases}.$$

Dyadic Odometer

The dyadic integers are given by

$$\mathbb{Z}_2 := \varprojlim_{k\to\infty} \mathbb{Z}/2^k\mathbb{Z}$$
$$\cong \{(a_k)_{k\in\mathbb{N}} \in \prod_{k\in\mathbb{N}} \{0,\dots,2^k-1\} |\ a_{k+1} \in \{a_k, a_k+2^k\}\}.$$

Equipped with pointwise addition they form a compact abelian group.
Define $\eta := (1,1,\dots) \in \mathbb{Z}_2$ and define $d : \mathbb{Z}_2 \to \mathbb{Z}_2$ by $d(a) := a + \eta$.Then $(\mathbb{Z}_2, d)$ is a minimal equicontinuous TDS called the **dyadic odometer**.

There is a factor map $\rho : Y \to \mathbb{Z}_2$, which is at most 2-to-1. In particular $\mathbb{Z}_2$ is the MEF of (Y,T) and (X,T).

Putting this together we have

$$\begin{array}{rcc} \text{Thue Morse subshift:} & (X,\sigma) & \text{5-equicontinuous} \\ \pi: & \downarrow & \\ \text{period doubling subshift:} & (Y,\sigma) & \text{3-equicontinuous} \\ \rho: & \downarrow & \\ \text{dyadic odometer:} & (\mathbb{Z}_2, d) & \text{(2-)equicontinuous.} \end{array}$$

In particular $(\mathbb{Z}_2, d)$ is the MEF of (X,σ) and (Y,σ). Also (Y,σ) is a 3-MEF of (X,σ).
Note that however the MEF of $(\{0,1\}^{\mathbb{Z}},\sigma)$ is trivial.

Thanks for your attention!

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